CS 1050: Constructing Proofs

Problem Set 1

Due on Wednesday, Sept 6, beginning of class

Problem 1

Direct Proofs (10 points)

1. Prove that for every real number x, $x^2 - 2x + 2 > 0$.
2. Suppose a and b are integers such that $a - b$ is divisible by 7. Prove that $a^2 - b^2$ is also divisible by 7. Prove that $a^3 - b^3$ is also divisible by 7.

Problem 2

Examples/Counter-examples (20 points)

Prove or disprove each of the following propositions by giving an example or a counter-example.

1. There is a natural number p such that $p, p+2, p+6$ and $p+8$ are all primes.
2. If x and y are positive irrational numbers, then $x \cdot y$ is also an irrational number.
3. If x and y are positive rational numbers, then x^y is also a rational number.
4. If A and B are two finite sets, then $|A \cup B| = |A| + |B|$. Here $|A|$ denotes the number of elements in the set A.

Problem 3

Proofs by Contrapositive (10 points)

1. Let x and y be real numbers. Prove that if $x + y$ is irrational, then either x or y is irrational.
2. Let a and b be integers. Prove that if $a \cdot b$ is odd, then a and b both are odd.

Problem 4

Proofs by Contradiction (10 points)

1. Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Let y be their average, i.e. $y = \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)$. Prove that $y \leq x_i$ for some $i \in \{1, 2, \ldots, n\}$.
2. Prove that for any integer n, at least one of the three integers $n, 2n+1, 4n+3$ is not divisible by 7.

Problem 5

Proofs by Cases (10 points)

1. For any real numbers x and y, prove that $|x + y| \leq |x| + |y|$. Here $|x|$ denotes the absolute value of x which equals x if $x \geq 0$ and equals $-x$ if $x < 0$. (*Hint: There are four cases depending on whether x and y are greater/less-or-equal to zero*).

2. If n is a natural number (written in usual decimal notation), prove that the last digit of n^4 is either $0, 1, 5$ or 6.

Problem 6

Puzzle (10 points)

There are seven jars with seven coins in each jar. In one of the jars, all coins are fake. All coins in all the remaining six jars are true coins. A fake coin looks exactly like a true coin but weighs less. A fake coin weighs 9 grams whereas a true coin weighs 10 grams. You are given a weighing scale. How will you find out which jar has fake coins with only one weighing?

Problem 7

Feedback (10 points)

1. How did you find this homework? Too easy? Too difficult?

2. Do you like solving mathematical puzzles?

3. Do you know (at least a little bit of) calculus?

4. Why are you taking this class? If it were not a required class, would you still take it?

5. How do you like the teaching so far? Too slow? Too fast? Any suggestions?