CS 1050B: Constructing Proofs

Problem Set 2 : Induction and Recursion

Due Friday, Sept 22nd, after the class

Problem 1 : Rosen 4.1: 3, 4, 17

1. Let $P(n)$ be the statement that $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for the positive integer n.

   a) Plugging in $n = 1$ we have that $P(1)$ is the statement $(1^2 = 1 \cdot 2 \cdot 3)/6$

   b) both sides of $P(1)$ shown in part (a) equal 1.

   c) $1^2 + 2^2 + \cdots + k^2 = k(k+1)(2k+1)/6$

   d) For the inductive step, we want to show that for each $k \geq 1$ that $P(k)$ implies $P(k+1)$. In other words, we want to show that assuming the inductive hypothesis (see part (c)) we can show
   $$1^2 + 2^2 + \cdots + k^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

   e)
   $$\begin{aligned}(1^2 + 2^2 + \cdots + k^2) + (k+1)^2 &= \frac{(k+1)(k+2)(2k+3)}{6} + (k+1)^2 \\ &= \frac{k+1}{6}(k(2k+1) + 6(k+1)) \\ &= \frac{k+1}{6}(2k^2 + 7k + 6) \\ &= \frac{(k+1)(k+2)(2k+3)}{6}\end{aligned}$$

2. Use induction to prove the statement that $1^3 + 2^3 + \cdots + n^3 = (\frac{n(n+1)}{2})^2$ for every positive integer n.

   a) Plugging in $n = 1$ we have that $P(1)$ is the statement $(1^3 = [1 \cdot (1+1)/2]^2$

   b) both sides of $P(1)$ shown in part (a) equal 1.

   c) $1^3 + 2^3 + \cdots + k^3 = (k(k+1)/2)^2$

   d)
   $$\begin{aligned}(1^3 + 2^3 + \cdots + k^3) + (k+1)^3 &= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\ &= (k+1)^2\left(\frac{k^2 + 4k + 4}{4}\right) \\ &= \left(\frac{(k+1)(k+2)}{2}\right)^2\end{aligned}$$

3. Use induction to prove the statement that $\sum_{j=1}^{n} j^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$ for every positive integer n.

   This proof follows the basic pattern of the previous proofs. The statement $P(n)$ that we wish to prove is
$$\sum_{j=1}^{n} j^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$
   where n is a positive integer. The basis step, $n = 1$, is true, since $1 \cdot 2 \cdot 3 \cdot 5/30 = 1$. Assume the displayed statement as the inductive hypothesis, and proceed as follows to prove $P(n+1)$:
$$\begin{aligned}(1^4 + 2^4 + \cdots + n^4) + (n+1)^4 &= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + (n+1)^4 \\ &= \frac{n+1}{30}\left(n(2n+1)(3n^2+3n-1) + 30(n+1)^3\right) \\ &= \frac{n+1}{30}(6n^4 + 39n^3 + 91n^2 + 89n + 30) \\ &= \frac{n+1}{30}(n+2)(2n+3)\left(3(n+1)^2 + 3(n+1) - 1\right)\end{aligned}$$

Problem 2 : Rosen 4.2: 4, 9

1. Let $P(n)$ be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for $n \geq 18$.

   a) Show statement $P(18)$, $P(19)$, $P(20)$, $P(21)$ are true, completing the basis step of the proof.
   $P(18)$: one 4-cent stamp and two 7-cent stamps.
   $P(19)$: three 4-cent stamps and one 7-cent stamp.
   $P(20)$: five 4-cent stamps.
   $P(21)$: three 7-cent stamps.

   b) What is the inductive hypothesis of the proof?
   The inductive hypothesis is the statement that using just 4-cent and 8-cent stamps we can form j cents postage for all j with $18 \leq j \leq k$, where we assume that $k \geq 21$.

   c) What do you need to prove in the inductive step?
   We must show, assuming the inductive hypothesis, that we can form $k+1$ cents postage using just 4-cent and 7-cent stamps.

   d) Complete the inductive step for $k \geq 21$.
   We want to form $k+1$ cents of postage. Since $k \geq 21$, we know that $P(k-3)$ is true, that is, that we can form $k-3$ cents of postage. Put one more 4-cent stamp on the envelope, and we have formed $k+1$ cents of postage, as desired.

2. Use strong induction to prove that $\sqrt{2}$ is irrational. [*Hint:* Let $P(n)$ be the statement that $\sqrt{2} \neq n/b$ for any positive integer b.]

Proof:

a) Following the hint, we let $P(n)$ be the statement that there is no positive integer b such that $\sqrt{2} = n/b$.

b) For the basis step, $P(1)$ is true because $\sqrt{2} > 1 \geq 1/b$ for all positive integers b.

c) For the inductive step, assume that $P(j)$ is ture for all $j \leq k$, where k is an arbitrary positive integer; we must prove that $P(k+1)$ is true.
So assume the contrary, that $\sqrt{2} = (k+1)/b$ for some positive integer b. Squaring both sides and clearing fractions, we have $2b^2 = (k+1)^2$. This tells us that $(k+1)^2$ is even, and so $k+1$ is even as well. Therefore we can write $k+1 = 2t$ for some positive integer t. Substituting, we have $2b^2 = 4t^2$, so $b^2 = 2t^2$. so b is even, so $b = 2s$ for some positive integer s. Then we have $\sqrt{2} = (k+1)/b = (2t)/(2s) = t/s$. But $t \leq k$, so this contradicts the inductive hypothesis, and our proof of the inductive step is complete.

Problem 3 : Rosen 4.3: 8, 45

1. Give a recursive definition of the sequence a_n, $n = 1, 2, 3, \ldots$ if

a) $a_n = 4n - 2$

$$a_1 = 2 \quad \text{and} \quad a_{n+1} = a_n + 4 \quad \text{for all} \quad n \geq 2$$

b) $a_n = 1 + (-1)^n$

$$a_1 = 0, \quad a_2 = 2, \quad \text{and} \quad a_n = a_{n-2} \quad \text{for all} \quad n \geq 3$$

c) $a_n = n(n+1)$

$$a_1 = 2 \quad \text{and} \quad a_n = a_{n-1} + 2n, \quad \text{or} \quad a_n = \frac{n+1}{n-1} a_{n-1} \quad \text{for all} \quad n \geq 2;$$

d) $a_n = n^2$

$$a_1 = 1 \quad \text{and} \quad a_n = a_{n-1} + 2n - 1, \quad \text{for all} \quad n \geq 2;$$

2. Use generalized induction to show that if $a_{m,n}$ is defined recursively by $a_{0,0} = 0$ and

$$a_{m,n} = \begin{cases} a_{m-1,n} + 1 & \text{if } n = 0 \text{ and } m > 0 \\ a_{m,n-1} + 1 & \text{if } n > 0 \end{cases}$$

then $a_{m,n} = m + n$ for all $(m, n) \in \mathbf{N} \times \mathbf{N}$

Proof:
The basis step requires that we show that this formula holds when $(m, n) = (0, 0)$. The

inductive step requires that we show that if the formula holds for all pairs smaller than (m, n) in the lexicographic ordering of $\mathbf{N} \times \mathbf{N}$, then it also holds for (m, n). for the basis step we have $a_{0,0} = 0 = 0 + 0$. For the inductive step, assume that $a_{m',n'} = m' + n'$ whenever (m', n') is less than (m, n) in the lexicographic ordering of $\mathbf{N} \times \mathbf{N}$. By the recursive definition, if $n = 0$ then $a_{m,n} = a_{m-1,n} + 1$; since $(m - 1, n)$ is smaller than (m, n), the inductive hypothesis tells us that $a_{m-1,n} = m - 1 + n$, so $a_{m,n} = m - 1 + n + 1 = m + n$, as desired. Now suppose that $n > 0$, so that $a_{m,n} = a_{m,n-1} + 1$. Again we have $a_{m,n-1} = m + n - 1$, so $a_{m,n} = m + n - 1 + 1 = m + n$, and the proof is complete.

Problem 4: Rosen 4.4: 15, 29, 30

1. Devise a recursive algorithm for computing the greatest common divisor of two nonnegative integers a and b with $a < b$ using the fact that $\gcd(a, b) = \gcd(a, b - a)$.

   procedure *gcd*(a, b : nonnegative integers with $a < b$)
   if $a = 0$ **then** $gcd(a, b) := b$
   else if $a = b - a$ **then** $gcd(a, b) := a$
   else if $a < b - a$ **then** $gcd(a, b) := gcd(a, b - a)$
   else $gcd(a, b) := gcd(b - a, a)$

2. Devise a recursive algorithm to find the nth term of the sequence defined by $a_0 = 1$, $a_1 = 2$, and $a_n = a_{n-1} \cdot a_{n-2}$, for $n = 2, 3, 4, \cdots$.

   procedure *sequence*(n : nonnegative integer)
   if $n < 2$ **then** $sequence(n) := n + 1$
   else $sequence(n) := sequence(n - 1) \cdot sequence(n - 2)$

3. Devise an iterative algorithm to find the nth term of the sequence defined in the above problem.

   procedure *iterative*(n : nonnegative integer)
   if $n = 0$ **then** $y := 1$
   else
   begin
   $x := 1$
   $y := 2$
   for $i := 1$ **to** $n - 1$
   begin
   $z := x \cdot y$
   $x := y$
   $y := z$
   end
   end
   {y is the n^{th} term of the sequence}