CS 1050B: Constructing Proofs

Quiz 1

1. (10 points) Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.

   The proof can be found on page 80 of Rozen. It is also closely related to the proof of problem 2.2 in Homework 2.

2. The factorial $n!$ is defined for a positive integer n as $n! \equiv n(n-1)\cdots 2 \cdot 1$.

   a) (8 points) Please give two different recursive definitions of the sequence $\{a_n\}$, $n = 1, 2, 3, \cdots$ if

   $$a_n = n!(n^2 + 2)$$

   Please show all of the important steps.

   b) (2 points) Which definition is better? Please briefly explain why that is the case.

   Anwser :

   a) Two possible definitions are

   i. $a_n = a_{n-1} + (n-1)!(n^3 - n^2 + 4n - 3)$ for $n \geq 2$.

   $$\begin{aligned} a_n &= n!(n^2+2) \\ a_{n-1} &= (n-1)!((n-1)^2+2) \\ a_n - a_{n-1} &= n!(n^2+2) - (n-1)!((n+1)^2+2) \\ &= n(n-1)!(n^2+2) - (n^2-2n+3)(n-1)! \\ &= (n-1)!(n(n^2+2) - (n^2-2n+3)) \\ &= (n-1)!(n^3-n^2+4n+3) \end{aligned}$$

   ii. $a_n = a_{n-1}\left(\frac{n^3+2n}{n^2-2n+3}\right)$ for $n \geq 2$.

   $$\begin{aligned} a_n &= n!(n^2+2) \\ a_{n-1} &= (n-1)!((n-1)^2+2) \\ a_n/a_{n-1} &= \frac{n!(n^2+2)}{(n-1)!((n-1)^2+2)} \\ &= \frac{n(n-1)!(n^2+2)}{(n-1)!(n^2-2n+3)} \\ &= \frac{n^3+2n}{n^2-2n+3} \end{aligned}$$

b) The second one is better since it does not involve calculating $(n-1)!$ each time.

3. (10 points) Use a merge sort to sort 4, 3, 2, 5, 1, 8, 7, 6. Show all the steps used by the algorithm. How many steps are required to merge the pair of lists at the last step before completion (This analysis should be similar to that one that is used in the last problem of the supplementary exercise).

   Anwser :
   We will show the tree and inverted tree that indicate how the sequence is taken apart and put back together.

   (4, 3, 2, 5, 1, 8, 7, 6)
   (4, 3, 2, 5) (1, 8, 7, 6)
   ((4, 3)(2, 5)) ((1, 8)(7, 6))
   ((3, 4)(2, 5)) ((1, 8)(6, 7))
   (2, 3, 4, 5) (1, 6, 7, 8)
   (1, 2, 3, 4, 5, 6, 7, 8)

   The number of comparisons is $m+n-r$, where the lists have m and n elements, respectively, and r is the number of elements remaining in one list at the point the other list is exhausted. Here, $m = n = 4$, so the answer is $8 - 3 = 5$ since the second list $(1, 6, 7, 8)$ has 3 elements when the first list has been emptied.

4. (10 points) Use induction to prove that

$$\sum_{a=1}^{n} \frac{1}{a(a+1)} = \frac{n}{n+1}$$

   for all $n \geq 1$.
   Hint: A similar infinite series $\sum_{a=1}^{\infty} \frac{1}{a(a+1)}$ is a typical telescoping series. It can be simplified in the following way

$$\begin{aligned}
\sum_{a=1}^{\infty} \frac{1}{a(a+1)} &= \sum_{a=1}^{\infty} (\frac{1}{a} - \frac{1}{a+1}) \\
&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots \\
&= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \cdots = 1
\end{aligned}$$

   Use this fact to construct your proof, notice that the series in the problem is a finite one; it is not same as the one given in the hint.

   Proof :
   The formula holds for $n = 1$, since $\sum_{a=1}^{1} \frac{1}{a(a+1)} = \frac{1}{1 \cdot 2} = 1/2 = \frac{1}{1+1}$; This is the base case.

Assume the inductive hypothesis that the formula holds for $n = k$,

$$\sum_{a=1}^{k} \frac{1}{a(a+1)} = \frac{k}{k+1}$$

We want to show that the formula also holds for $n = k + 1$.

$$\begin{aligned}
\sum_{a=1}^{k+1} \frac{1}{a(a+1)} &= \sum_{a=1}^{k+1} \left(\frac{1}{a} - \frac{1}{a+1}\right) \\
&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{k} - \frac{1}{k+1}\right) + \left(\frac{1}{k+1} - \frac{1}{k+2}\right) \\
&= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \cdots + \left(-\frac{1}{k+1} + \frac{1}{k+1}\right) - \frac{1}{k+2} \\
&= 1 - \frac{1}{k+2} \\
&= \frac{k+1}{k+2}
\end{aligned}$$

Q.E.D.