CS 1050B: Constructing Proofs

Supplementary Exercises 1 : Proof, Induction, and Recursion

Answer Key

Problem 1 : Rosen Chapter 1 Supplementary Exercises

1. Problem 4

   a) converse: If I drive to work today, then it will rain.
   contrapositive: If I do not drive to work today, then it will not rain.
   inverse: If it does not rain today, then I will not drive to work.

   b) converse: If $x \geq 0$ then $|x| = x$.
   contrapositive: If $x < 0$ then $|x| \neq x$.
   inverse: If $|x| \neq x$, then $x < 0$.

   c) converse: If n^2 is greater than 9, then n is greater than 3.
   contrapositive: If n^2 is not greater than 9, then n is not greater than 3.
   inverse: If n is not greater than 3, then n^2 is not greater than 9.

2. Problem 26

   a) It will snow today, but I will not go skiing tomorrow.
   b) Some person in this class does not understand mathematical induction.
   c) All students in this class like discrete mathematics.
   d) There is some mathematics class in which all the students stay awake during lectures.

3. Problem 32
   Proof by contraposition:
   If x is rational, then $x = p/q$ for some integers p and q with $q \neq 0$. Then $x^3 = p^3/q^3$, and we have expressed x^3 as the quotient of two integers, the second of which is not zero. This by definition means that x^3 is rational, and that completes the proof of the contrapositive of the original statement.

4. Problem 33
   We give a proof by contraposition that if $\sqrt{x}$ is rational, then x is rational, assuming throughout that $x \geq 0$. Suppose that $\sqrt{x} = p/q$ is rational, $q \neq 0$. Then $x = (\sqrt{x})^2 = p^2/q^2$ is also rational.

5. Problem 34
   Let m be the square root of n, rounded down if it is not a whole number. (In the notation to be introduced in Section 2.3, we are letting $m = \lfloor\sqrt{n}\rfloor$.) Different choices of m correspond to a partition of $\mathbf{N}$, namely into $\{0\}$, $\{1, 2, 3\}$, $\{4, 5, 6, 7, 8\}$,.... So every n is in exactly one of these sets.

Problem 2 : Rosen Chapter 4 Supplementary Exercises

1. Problem 2
   The proposition is true for $n = 1$, since $1^3 + 3^3 = 28 = 1(1 + 1)^2(2 \cdot 1^2 + 4 \cdot 1 + 1)$. Assume the inductive hypothesis. Then

   $$\begin{aligned} 1^3 + 3^3 + \cdots + (2n+1)^3 + (2n+3)^3 &= (n+1)^2(2n^2 + 4n + 1) + (2n+3)^3 \\ &= 2n^4 + 8n^3 + 11n^2 + 6n + 1 + 8n^3 + 37n^2 + 54n + 27 \\ &= 2n^4 + 17n^3 + 47^2 + 60n + 28 \\ &= (n+2)^2(2n^2 + 8n + 7) \\ &= (n+2)^2(2(n+1)^2 + 4(n+1) + 1) \end{aligned}$$

2. Problem 6
   We prove this statement by induction. The base case is $n = 5$, and indeed $5^2+5 = 30 < 32+2^5$. Assuming the inductive hypothesis, we have $(n+1)^2 + (n+1) = n^2 + 3n + 2 < n^2 + 4n < n^2 + n^2 + 2n^2 < 2(n^2 + n)$, which is less than $2 \cdot 2^n$ by the inductive hypothesis, and this equals $2^{(}n + 1)$, as desired.

3. Problem 9
   Let $P(n)$ be the statement that $a-b$ is a factor of $a^n - b^n$. We want to show that $P(n)$ is true for all positive integers n, and of course we will do so by induction. If $n = 1$, then we have the trivial statement that $a - b$ is a factor of $a - b$. Next assume that inductive hypothesis, that $P(n)$ is true. We want to show $P(n + 1)$, that $a - b$ is a factor of $a^{n+1} - b^{n+1}$. The trick is to rewrite $a^{n+1} - b^{n+1}$ by subtracting and adding ab^n. We obtain $a^{n+1} - b^{n+1} = a^{n+1} - ab^n + ab^n - b^{n+1} = a(a^n - b^n) + b^n(a - b)$. Now this expression contains two terms. By the inductive hypothesis, $a - b$ is a factor of the first term and is a factor of the second. Therefore $a - b$ is a factor of the entire expression, and we are done.

4. Problem 43
   The basis step is wrong. The statement makes no sense for $n = 1$, since the last term on the left-hand side would then be $1/(0 \cdot 1)$, which is undefined. The first n for which it makes sense is $n = 2$, when it reads

   $$\frac{1}{1 \cdot 2} = \frac{3}{2} - \frac{1}{2}$$

   Of course this statement is false, since $\frac{1}{2} \neq 1$. Therefore the basis step fails, and so the "theorem" is not true.

Problem 3 : Rozen

1. Exercise 4.2 : 7
   We claim that we can form all amounts of money greater than or equal to 5 dollars. Let $P(n)$ be the statement that we can form n dollars using just 2-dollar and 5-dollar bills. We want to prove that $P(n)$ is true for all $n \geq 5$. We observe that the basis step is true for $n = 5$ and 6.

assume the inducive hypothesis, that $P(j)$ is true for all j with $5 \leq j \leq k$, where k is a fixed integer greater than or equal to 6. We want to show that $P(k+1)$ is true. Because $k-1 \geq 5$, we know that $P(k-1)$ is true, that is, that we can form $k-1$ dollars. Add another 2-dollar bill, and we have formed $k+1$ dollars, as desired.

2. Exercise 4.2 : 8

   To prove this by strong induction, let $P(n)$ be the statement that we can form $5n$ dollars in gift certificates using just 25-dollar and 40-dollar certificates. We want to prove that $P(n)$ is true for all $n \geq 28$. It's easy to show that $P(n)$ is true for $n = 28, 29, 30, 31, 32$. Assume the inductive hypothesis, that $P(j)$ is true for all j with $28 \leq j \leq k$, where k is a fixed integer greater than or equal to 32. We want to show that $P(k+1)$ i s true. Because $k-4 \geq 28$, we know that $P(k-4)$ is true, that is, that we can form $5(k-4)$ dollars. Add one more 25-dollar certificate, and we have formed $5(k+1)$ dollars, as desired.

3. Exercise 4.3 : 9

   $F(1) = 1$ and $F(n+1) = F(n) + n + 1$

4. Exercise 4.3 : 13

   We prove this using the principle of mathematical induction. The base case is $n = 1$, and in that case the statement to be proved is just $f_1 = f_2$; this is true since bothe values are 1. Next we assume the inductive hypothesis, that

   $$f_1 + f_3 + \cdots + f_{2n-1} = f_{2n},$$

   then we have

   $$\begin{aligned} f_1 + f_3 + \cdots + f_{2n-1} + f_{2n+1} &= f_{2n} + f2n + 1 \qquad \text{(by the inductive hypothesis)} \\ &= f_{2n+2} \qquad \text{(by the definition of the Fibonacci numbers).} \end{aligned}$$

5. Exercise 4.3 : 25

   a) $0 \in S$; and if $x \in S$, then $x + 2 \in S$ and $x - 2 \in S$.

   b) $2 \in S$; and if $x \in S$, then $x + 3 \in S$.

   c) $1 \in S$, $2 \in S$, $3 \in S$, and $4 \in S$, and if $x \in S$, then $x + 5 \in S$.

6. Exercise 4.4 : 23

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procedure $square(n$ : nonnegative integer)
if $n = 0$ then $square(n) := 0$
else $square(n) := square(n-1) + 2(n-1) + 1$
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   The proof of correctness, by mathematical induction, practically writes itself as well. Let $P(n)$ be the statement that this algorithm correctly computes n^2. Since $0^2 = 0$, the algorithm works correctly (using the **if** clause) if the input is 0, Assume that the algorithm works correctly for input k. Then for input $k+1$ it gives as output (because of the **else** clause) its output when the input is k, plus $2(k+1-1)+1$. By the inductive hypothesis, its output as k is k^2, so its output as $k+1$ is $k^2+2(k+1-1)+1=k^2+2k+1=(k+1)^2$.

7. Exercise 4.4 : 46

   From the analysis given before the statement of Lemma 1, it follows that the number of comparisons is $m+n-r$, where the lists have m and n elements, respectively, and r is the number of elements remaining in one list at the point the other list is exhausted. In this exercise $m=n=5$, so the answer is always 10-r.

   a) 9, since the second list has only 1 element when the first list has been emptied.

   b) 5, since the second list has 5 elements when the first list has been emptied.

   c) 8, since the second list has 2 elements when the first list has been emptied.