CS 1050B: Constructing Proofs

Supplementary Exercises 2 : Counting, Discrete Probability

Answer Key

Section 1 : Rosen Chapter 7

1. 7.1 Problem 6

   a) $a_n = a_{n-1}$
   b) $a_n - a_{n-1} = 2 \Rightarrow a_n = a_{n-1} + 2$
   c) $a_n = a_{n-1} + 2$
   d) $a_n = 5a_{n-1}$
   e) $a_n - a_{n-1} = n^2 - (n-1)^2 = 2n - 1 \Rightarrow a_n = a_{n-1} + 2n - 1$
   f) $a_n = a_{n-1} + 2n$
   g) $a_n - a_{n-1} = n + (-1)^n - (n-1) - (-1)^{n-1} = 1 + 2(-1)^n \Rightarrow a_n = a_{n-1} + 1 + 2(-1)^n$
   h) $a_n = na_{n-1}$

2. Supplementary Exercises 8

   If we add the equations, we obtain $a_n + b_n = 2a_{n-1}$, which means that $b_n = 2a_{n-1} - a_n$. If we now substitute this back into the first equation, we have $a_n = a_{n-1} + (2a_{n-2} - a_{n-1}) = 2a_{n-2}$. The initial conditions are $a_0 = 1$ and $a_1 = 3$. It is clear that $a_{2n} = 2^n a_0 = 2^n$, and $a_{2n+1} = 2^n a_1 = 3 \cdot 2^n$. We also need a formula for b_n, of course. From $b_n = 2a_{n-1} - a_n$, we have $b_{2n} = 3 \cdot 2^n - 2^n = 2^{n+1}$, and $b_{2n+1} = 2 \cdot 2^n - 3 \cdot 2^n = -2^n$.

Section 2 : Rosen Chapter 5 Supplementary Exercises

1. Problem 6
   $9 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 90,000$

2. Problem 8

   a) All the integers from 100 to 999 have three decimal digits, and there are $999 - 100 + 1 = 900$ of these.
   b) In addition to the 900 three digit numbers, there are 9 one-digit positive integers, for a total of 909.

c) There is 1 one-digit number with a 9. Among the two-digit numbers, there are the 10 numbers from 90 to 99, together with the 8 numbers $19, 29, \cdots, 89$, for a total of 18. Among the three-digit numbers, there are the 100 from 900 to 999 and there are, for each century from the 100's to the 800's, again $1 + 18 = 19$ numbers with at least 9; this gives a total of $100 + 8 \cdot 19 = 252$. Thus our final answer is $1 + 18 + 252 = 271$. Alternatively, we can compute this as $10^3 - 9^3 = 271$, since we want to subtract from the number of three-digit nonnegative numbers (with leading 0's allowed) the number of those that use only the nine digits 0 through 8.

d) Since we can use only even digits, there are $5^3 = 125$ ways to specify a three-digit number, allowing leading 0's. Since, however, the number $0 = 000$ is not in our set, we need to subtract 1, obtain the answer 124.

e) The numbers in question are either of the form $d55$ or $55d$, with $d \neq 5$, or 555. Since d can be any of nine digits, there are $9 + 9 + 1 = 19$ such numbers.

f) All 9 one-digit numbers are palindromes. The 9 two-digit numbers $11, 22, \cdots, 99$ are palindromes. For three-digit numbers, the first digit (which must equal the third digit) can be any of the 9 nonzero digits, and the second digit can be any of the 10 digits, giving $9 \cdot 10 = 90$ possibilities. Therefore the answer is $9 + 9 + 90 = 108$.

3. Problem 10
   Using the generalized pigeonhole principle, we see that we need $5 \times 12 + 1 = 61$ people.

4. Problem 12
   There are $7 \times 12 = 84$ day-month combinations. Therefore we need 85 people to ensure that two of them were born on the same day of the week and in the same month.

Section 3 : Rosen Chapter 6 Supplementary

1. Problem 4
   The denominator of each probability is the number of 7-card poker hands, namely $C(52, 7) = 133784560$

   a) The number of such hands is $13 \cdot 12 \cdot 4$, since there are 13 ways to choose the kind for the four, then 12 ways to choose another kind for the three, then $C(4, 3) = 4$ ways to choose which three cards of that second kind to use. Therefore the probability is $624/133784560$.

   b) The number of such hands is $13 \cdot 4 \cdot 66 \cdot 6^2$, since there are 13 ways to choose the kind for the three, $C(4, 3) = 4$ ways to choose which three cards of that kind to use, then $C(12, 2) = 66$ ways to choose two more kinds for the pairs, then $C(4, 2) = 6$ ways to choose which two cards of each of those kinds to use. Therefore the probability is $123552/133784560$.

   c) The number of such hands is $286 \cdot 6^3 \cdot 10 \cdot 4$, since there are $C(13, 3) = 286$ ways to choose the kinds for the pairs, $C(4, 2) = 6$ ways to choose which two cards of each of those kinds to use, 10 ways to choose the kind for the singleton, and 4 ways to choose which card of that kind to use. Therefore the probability is $2471040/133784560$.

d) The number of such hands is $78 \cdot 6^2 \cdot 165 \cdot 4^3$, since there are $C(13,2) = 78$ ways to choose the kinds for the pairs, $C(4,2) = 6$ ways to choose which two cards of each of those kinds to use, $C(11,3) = 165$ ways to choose the kind for the singletons, and 4 ways to choose which card of each of those kinds to use. Therefore the probability is 29652480/133784560.

e) The number of such hands is $1716 \cdot 4^7$, since there are $C(13,7) = 1716$ ways to choose the kinds and 4 ways to choose which card of each of kind to use. Therefore the probability is 28114944/133784560.

f) The number of such hands is $4 \cdot 1716$, since there are 4 ways to choose the suit for the flush and $C(13,7) = 1716$ ways to choose the kinds in that suit. Therefore the probability is 6864/133784560.

g) The number of such hands is $8 \cdot 4^7$, since there are 8 ways to choose the kind for the straight to start at ($A, 2, 3, 4, 5, 6, 7$, or 8) and 4 ways to choose the suit for each kind. Therefore the probability is 131072/133784560.

h) There are only $4 \cdot 8$ straight flushes, since the only choice is the suit and the starting kind (see part (g)). Therefore the probability is 32/133784560.

2. Problem 6

a) Each of the outcomes 1 through 12 occurs with probability 1/12, so the expectation is $(1/12)(1 + 2 + 3 + \cdots + 12) = 13/2$.

b) We compute $V(X) = E(X^2) - E(X)^2 = (1/12)(1^2 + 2^2 + \cdots + 12^2) - (13/2)^2 = 143/12$.

3. Problem 8

a) Since expected value is linear, the expected value of the sum is the sum of the expected values, each of which is 13/2 by Exercise 6a. Therefore the answer is 13.

b) Since variance is linear for independent random variables, and clearly these variables are independent, the variance of the sum is the sum of the variances, each of which is 143/12 by Exercise 6b. Therefore the answer 143/6.

4. Problem 10

a) Since expected value is linear, the expected value of the sum is the sum of the expected values, which are 9/2 by Exercise 5a and 13/2 by Exercise 6a. Therefore the answer is $(9/2) + (13/2) = 11$.

b) Since variance is linear for independent random variables, and clearly these variables are independent, the variance of the sum is the sum of the variances, which are 21/4 by Exercise 5b and 143/12 by Exercise 6b. Therefore the answer is $(21/4) + (143/12) = 103/6$.