CS 1050B: Constructing Proofs

Supplementary Exercises 3 : Comprehensive

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Rosen Chapter 1 : Logic and Proofs

1. Chapter 1 Supplementary P.40
   Assuming the truth of the theorem that states that $\sqrt{n}$ is irrational whenever n is a positive integer that is not a perfect square, prove that $\sqrt{2}+\sqrt{3}$ is irrational.

   Solution: We give a proof by contradiction. If $\sqrt{2}+\sqrt{3}$ were rational, then so would be its square, which is $5+2\sqrt{6}$. Subtracting 5 and dividing by 2 then shows that $\sqrt{6}$ is rational, but this contradicts the theorem we are told to assume.

2. 1.7 P.34
   Prove that between every two rational numbers there is an irrational number.

   Solution: The average of two different numbers is certainly always between the two numbers. Furthermore, the average a of a rational number x and irrational number y must be irrational, because the equation $a=(x+y)/2$ leads to $y=2a-x$, which would be rational if a were rational.

Chapter 4 : Induction and Recursion
Chapter 7 : Recurrence Relations

3. Chapter 4 Supplementary P.4
   Use mathematical induction to show that

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$$

   whenever n is a positive integer.

   Solution: Proof by induction
   This is true for $n=1$, since $1/3=1/3$. Under the inductive hypothesis

$$\begin{aligned}
\frac{1}{1\cdot 3}+\cdots+\frac{1}{(2n-1)(2n+1)}+\frac{1}{(2n+1)(2n+3)} &= \frac{n}{2n+1}+\frac{1}{(2n+1)(2n+3)} \\
&= \frac{1}{2n+1}\left(n+\frac{1}{2n+3}\right) \\
&= \frac{1}{2n+1}\left(n+\frac{(2n+1)(n+1)}{2n+3}\right) \\
&= \frac{n+1}{2n+3}
\end{aligned}$$

4. 4.2 P.12
   Use strong induction to show that every positive integer n can be written as a sum of distinct power of two, that is, as a sum of a subset of the integers $2^0 = 1, 2^1 = 2, 2^2 = 4$, and so on. [*Hint*:For the inductive step, separately consider the case where $k+1$ is even and where it is odd. When it is even, note that $(k+1)/2$ is an integer.]

   Solution: Proof by strong induction
   The basis step is to note that $1 = 2^0$. Notice for subsequent steps that $2 = 2^1$, $3 = 2^1 + 2^0$, $4 = 2^2$, $5 = 2^2 + 2^0$, and so on. Indeed this is simply the representation of a number in binary form (base two). Assume the inductive hypothesis, that every positive integer up to k can be written as a sum of distinct powers of 2. We must show that $k+1$ can be written as a sum of distinct powers of 2. If $k+1$ is odd, then k is even, so 2^0 added. If $k+1$ is even, then $(k+1)/2$ is a positive integer, so by the inductive hypothesis $(k+1)/2$ can be written as a sum of distinct powers of 2. Increasing each exponent by 1 doubles the value and gives us the desired sum for $k+1$.

5. Give a recursive definition of the sequence a_n, $n = 1, 2, 3, \cdots$ if

   a) $a_n = 5$
   b) $a_n = 10^n$
   c) $a_n = n + (-1)^n$
   d) $a_n = n!(n^2 + 2)$
      (This was the first problem from quiz 1. Check out the solution if you are stuck. We are expecting you to be able to write the solution that way in the final exam.)

   Solution:

   a) $a_1 = 5 \quad \text{and} \quad a_n = a_{n-1} \quad \text{for all} \quad n \geq 1$
      $a_n - a_{n-1} = 0 \Rightarrow a_n = a_{n-1}$
   b) $a_1 = 10 \quad \text{and} \quad a_n = 10a_{n-1} \quad \text{for all} \quad n \geq 1$
      $a_n/a_{n-1} = 10 \Rightarrow a_n = 10a_{n-1}$
   c) $a_1 = 0 \quad \text{and} \quad a_n = a_{n-1} + 1 + 2(-1)^n \quad \text{for all} \quad n \geq 1$
      $a_n - a_{n-1} = n + (-1)^n - (n-1) - (-1)^{n-1} = 1 + 2(-1)^n \Rightarrow a_n = a_{n-1} + 1 + 2(-1)^n$
   d) Here are two possible recursive definitions out of the infinitely many possible definitions.
      i. $a_1 = 3 \quad \text{and} \quad a_n = a_{n-1} + (n-1)!(n^3 - n^2 + 4n - 3)$ for all $n \geq 2$.

$$\begin{aligned}
a_n &= n!(n^2 + 2) \\
a_{n-1} &= (n-1)!((n-1)^2 + 2) \\
a_n - a_{n-1} &= n!(n^2 + 2) - (n-1)!((n+1)^2 + 2) \\
&= n(n-1)!(n^2 + 2) - (n^2 - 2n + 3)(n-1)! \\
&= (n-1)!(n(n^2 + 2) - (n^2 - 2n + 3)) \\
&= (n-1)!(n^3 - n^2 + 4n + 3)
\end{aligned}$$

ii. $a_1 = 3$ and $a_n = a_{n-1}\left(\frac{n^3+2n}{n^2-2n+3}\right)$ for all $n \geq 2$.

$$\begin{aligned} a_n &= n!(n^2+2) \\ a_{n-1} &= (n-1)!((n-1)^2+2) \\ a_n/a_{n-1} &= \frac{n!(n^2+2)}{(n-1)!((n-1)^2+2)} \\ &= \frac{n(n-1)!(n^2+2)}{(n-1)!(n^2-2n+3)} \\ &= \frac{n^3+2n}{n^2-2n+3} \end{aligned}$$

6. Find the solution to each of these recurrence relations and initial conditions. Use an iterative approach

   a) $a_n = a_{n-1} + 2n + 3, a_0 = 4$
   b) $a_n = 2a_{n-1} - 1, a_0 = 1$
   c) $a_n = 2na_{n-1}, a_0 = 1$

   Solution:

   a) $a_n = n^2 + 4n + 4$ for $n \geq 1$

$$\begin{aligned} a_n &= 3 + 2n + a_{n-1} \\ &= 3 + 2n + (3 + 2(n-1) + a_{n-2}) = (2 \cdot 3 + 2n + 2(n-1)) + a_{n-2} \\ &= (2 \cdot 3 + 2n + 2(n-1)) + (3 + 2(n-2) + a_{n-3}) \\ &= (3 \cdot 3 + 2n + 2(n-1) + 2(n-2)) + a_{n-3} \\ &\vdots \\ &= (n \cdot 3 + 2n + 2(n-1) + 2(n-2) + \cdots + 2(n-(n-1))) + a_{n-n} \\ &= (n \cdot 3 + 2n + 2(n-1) + \cdots + 2 \cdot 1) + a_0 \\ &= 3n + 2 \cdot \frac{n(n+1)}{2} + 4 = n^2 + 4n + 4. \end{aligned}$$

   b) $a_n = 1$ for $n \geq 1$

$$\begin{aligned} a_n &= -1 + 2a_{n-1} \\ &= -1 + 2(-1 + 2a_{n-2}) = -3 + 4a_{n-2} \\ &= -3 + 4(-1 + 2a_{n-3}) = -7 + 8a_{n-3} \\ &\vdots \\ &= -(2^n - 1) + 2^n a_{n-n} = -2^n + 1 + 2^n \cdot 1 = 1 \end{aligned}$$

c) $a_n = 2^n n!$ for $n \geq 1$

$$\begin{aligned}
a_n &= 2na_{n-1} \\
&= 2n(2(n-1)a_{n-2} = 2^2(n(n-1))a_{n-2} \\
&= 2^2(n(n-1))(2(n-2))a_{n-3} = 2^3(n(n-1)(n-2))a_{n-3} \\
&\vdots \\
&= 2^n n(n-1)(n-2)(n-3)\cdots(n-(n-1))a_{n-n} \\
&= 2^n n(n-1)(n-2)\cdots 1 \cdot a_0 \\
&= 2^n n!
\end{aligned}$$

7. 7.3 P.17

Suppose that the votes of n people for different candidates (where there can be more than two candidates) for a particular office are the elements of a sequence. A person wins the election if this person receives a majority of the votes.

a) Devise a divide-and-conquer algorithm that determines whether a candidate received a majority and, if so, determine who this candidate is. [*Hint*: Assume that n is even and split the sequence of votes into two sequences, each with $n/2$ elements. Note that a candidate could not have received a majority of votes without receiving a majority of votes in at least one of the two halves.]

b) Use the Mater Theorem to estimate the number of comparisons needed by the algorithm you devised in part 1.

Solution:

a) Our recursive algorithm will take a sequence of names and determine whether one name occurs as more than half of the elements of the sequence, and if so, which name that is. If the sequence has just one element, then the one person on the list is the winner. For the recursive step, divide the list into two parts– the first half and the second half– as equally as possible. As is pointed out in the hint, no one could have gotten a majority of the votes on this list without having a majority in one half or the other, since if a candidate got less than or equal to half the votes in each half, then he got less than or equal to half the votes in all (this is essentially just the distributive law). Apply the algorithm recursively to each half to come up with at most two names. Then run through the entire list to count the number of occurrences of each of those names to decide which, if either, is the winner. this requires at most $2n$ additional comparisons for a list of length n.

b) We apply the Master Theorem with $a = 2, b = 2, c = 2$, and $d = 1$. Since $a = b^d$, we know that the number of comparisons is $O(n^d \log n) = O(n \log n)$.

Chapter 5 : Counting

Chapter 6 : Discrete Probability

8. Chapter 5 Supplementary P.18
Show that if five points are picked in the interior of a square with a side length of 2, then at least two of these points are no farther than $\sqrt{2}$ apart.

Solution: Divide the interior of the square, with lines joining the midpoints of opposite sides, into four 1×1 square. By the pigeonhole principle, at least two of the five points must be in the same small square. The furthest apart two points in a square could be is the length of the diagonal, which is $\sqrt{2}$ for a square 1 unit on the side.

9. Chapter 5 Supplementary P.34
How many different arrangements are there of either people seated at a round table, where two arrangements are considered the same if one can be obtained from the other by a rotation?

Solution: We can assume that the first person sits in the northernmost seat (thus eliminating the rotation ambiguity.) Then there are $P(7,7)$ ways to seat the remaining people, since they form a permutation reading clockwise from the first person. Therefore the answer is $7! = 5040$.

10. Chapter 6 Supplementary P.4
This problem appeared in the second supplementary set. If you have not done this problem, I strongly recommend you to finish it by yourself.

Solution: See supplementary 2 solution key.

11. Let's play cards

a) How many ways are there to pick out 5 cards (a 5-card hand) in a 52-card deck?

b) How many 5-card hands are there that contains no pairs?

c) What is the probability that a 5-card hand contains no pairs?

Solution:

a) $C(52,5)$

b) $C(13,5)4^5 = \frac{13!}{5!8!} \cdot 4^5$
Choose 5 kinds out of 13, for each kind, there are 4 ways to specify the suit. There is another way to get to this solution, $\frac{C(13,1)C(4,1)C(12,1)C(4,1)C(11,1)C(4,1)C(10,1)C(4,1)C(9,1)C(4,1)}{P(5,5)}$.
You can verify that those two expressions are the same.

c) $C(13,5)4^5/C(52,5)$

12. What is the expected value when a \$ 1 lottery ticket is bought in which the purchaser wins exactly \$ 10 million if the ticket contains the six winning numbers chosen from the set $\{1, 2, 3, \cdots, 50\}$ and the purchaser wins nothing otherwise?

Solution:
There are $C(50,6)$ equally likely possible outcomes when the state picks its winning numbers. The probability of winning \$10 million is therefore $1/C(50,6)$, and the probability of winning

\$0 is $1-(1/C(50,6))$. by definition, the expectation is therefore $10000000\cdot 1/C(50,6)+0=10000000/15890700\approx 0.63$.

Rosen Chapter 3 : Number Theory

13. homework 8.