A Combinatorial Procedure for Finding Isolating Neighbourhoods and Index Pairs

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Synopsis. A purely combinatorial procedure for finding an isolating neighbourhood and an index pair contained in a given set being a finite union of cubes in $\mathbf{R}^s$ is presented. It is applied for a computer-assisted computation of the Conley index of an isolated invariant subset of the Henon attractor. As a corollary, it is shown that the Henon attractor contains periodic orbits of all principal periods except for 3 and 5.

Key words: Isolating neighbourhood, isolated invariant set, index pair, Conley index.

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1. Introduction

The Conley index has already found a number of applications in the theory of dynamical systems. It can serve as a tool for detection of stationary points, connecting trajectories, periodic orbits and chaos. Being a purely topological tool, it does not take into account the information carried by derivatives of dynamical systems. Therefore, the description of the dynamics provided by it is generally less accurate than the one provided by the smooth methods. Paradoxically, the topological character of the Conley index decides about its strength. This is because its computation is simpler since it need not involve derivatives. It is even possible to compute the index rigorously without the detailed knowledge of the dynamical system. The computer assisted proof of the existence of chaotic dynamics in the Lorenz Equations (with nonclassical parameters) given by K.Mischaikow and M.Mrozek in [3] and [4] rests on this very feature of the Conley index. Apart from the calculation of a Poincare map arising from the Lorenz Equations and finding bounds for discretization and rounding errors, the role of a computer in their reasoning was to perform a check that a given set is an isolating block, build a finitely representable index pair for its invariant part and then compute the homomorphism induced by the index map of the quotient space of the index pair on cohomology. The aim of this paper is to show how the search for both an isolating neighbourhood and an index pair can be performed by a computer itself. This is meant to be a step towards automatizing computer-assisted investigation of concrete dynamical systems. The algorithms described in this paper, combinatorial methods of algebraic topology and the Conley index theory for decompositions of isolated invariant sets (see [9]), collected together can make a general method of reducing the problem of describing the dynamics of isolated invariant sets in terms of semiconjugacies to elementary problems of algebraic nature. This approach should be particularly useful for the computer-assisted investigation of chaotic invariant sets. In the last section we present the result of a computation of the Conley index of a decomposition of an isolated invariant set of the Henon map whose isolating neighbourhood and index pair was found using this technique. As a corollary, we show that the Henon attractor contains periodic trajectories of all periods except for 3 and 5.

2. Notation and statement of results

Fix $s \in \mathbf{N}$ and positive real numbers d_i, $i = 1, 2, \ldots, s$. Put

$$\Omega = \Big\{ \prod_{i=1}^{s} [k_i d_i, (k_i + 1) d_i] \ : \ k_i \in \mathbf{Z},\ i = 1, 2, \ldots, s \Big\}.$$

In what follows, we shall mainly deal with subsets of $\mathbf{R}^s$ being unions of members of Ω. Such sets will be called representable. For a set $\mathcal{B} \subset \Omega$ by $|\mathcal{B}|$ we denote the union of all members of $\mathcal{B}$. Throughout this paper, f will denote a fixed continuous map of $\mathbf{R}^s$ into itself. A fixed function $\mathcal{F} : \Omega \to 2^{\Omega}$ is assumed to satisfy the following condition

$$\forall_{K \in \Omega} \quad f(K) \subset \operatorname{int}\ |\mathcal{F}(K)|. \tag{2.1}$$

One can easily see that the above condition is equivalent to the following one

$$\forall_{x \in \mathbf{R}^n} \quad \forall_{K, L \in \Omega} \quad x \in K \text{ and } f(x) \in L \Rightarrow L \in \mathcal{F}(K). \tag{2.2}$$

For a set $\mathcal{B} \subset \Omega$ we put (cf [2]):

$$\mathcal{F}(\mathcal{B}) = \bigcup_{K \in \mathcal{B}} \mathcal{F}(K),$$

$$\mathcal{F}^{-1}(\mathcal{B}) = \Big\{ K \in \Omega : \mathcal{F}(K) \cap \mathcal{B} \neq \emptyset \Big\},$$

$$\mathrm{o}(\mathcal{B}) = \Big\{ K \in \Omega : K \cap |\mathcal{B}| \neq \emptyset \Big\},$$

$$\mathrm{d}(\mathcal{B}) = o(\mathcal{B}) \setminus \mathcal{B},$$

$$\begin{aligned} \mathrm{Inv}_{\mathcal{F}}(\mathcal{B}) = \ & \Big\{ K \in \Omega : \text{ there exists a double-sided sequence} \\ & \{K_n\}_{n \in \mathbf{Z}} \subset \mathcal{B} \text{ such that } K_0 = K \text{ and} \\ & K_{n+1} \in \mathcal{F}(K_n) \text{ for all } n \in \mathbf{Z} \Big\}. \end{aligned}$$

Notice that $\mathcal{F}$ provides upper bounds for the image and the inverse image of a given representable set $N = |\mathcal{B}|$, $\mathcal{B} \subset \Omega$ under the map f. They are given by $\mathrm{int}|\mathcal{F}(\mathcal{B})|$ and $\mathrm{int}|\mathcal{F}^{-1}(\mathcal{B})|$. For a set $N \subset \mathbf{R}^s$ by $\mathrm{Inv}_f(N)$ we shall denote the maximal invariant subset of N with respect to the map f (see [2],[6]).

The following proposition gives a relationship between $\mathrm{Inv}_{\mathcal{F}}$ and Inv_f. Like all other statements below, is proved in the next section.

Proposition 2.1. *For each* $\mathcal{B} \subset \Omega$, $\mathrm{Inv}_f|\mathcal{B}| \subset |\mathrm{Inv}_{\mathcal{F}}(\mathcal{B})|$.

Below we state four theorems, which provide a combinatorial procedure for constructing isolating neighbourhoods and index pairs. A method of calculating $\mathrm{Inv}_{\mathcal{F}}(\mathcal{B})$ for a finite set $\mathcal{B} \subset \Omega$ is given by the following easy theorem.

Theorem 2.1. *Let* $\mathcal{B}$ *be a finite subset of* Ω*. Define inductively:*

$$\begin{aligned} &\mathcal{B}_0 = \mathcal{B}, \\ &\mathcal{B}_{n+1} = \mathcal{F}(\mathcal{B}_n) \cap \mathcal{B}_n \cap \mathcal{F}^{-1}(\mathcal{B}_n) \text{ for all } n \in \mathbf{Z}^+. \end{aligned}$$

There exists an $n_0 \in \mathbf{Z}^+$ *such that for all* $n \geq n_0$, $\mathcal{B}_n = \mathcal{B}_{n_0}$*. For such* n_0, $\mathcal{B}_{n_0} = Inv_{\mathcal{F}}(\mathcal{B})$.

There exists a simple, purely combinatorial condition which guarantees that a given representable set is an isolating neighbourhood for f.

Theorem 2.2. *Let* $\mathcal{B}$ *be a finite subset of* Ω*. If* $\mathrm{Inv}_{\mathcal{F}}(\mathrm{o}(\mathcal{B})) \subset \mathcal{B}$ *then* $|\mathcal{B}|$ *is an isolating neighbourhood for* f.

The next theorem provides a method of constructing a representable isolating neighbourhood.

Theorem 2.3. *Let* $\mathcal{B}$ *be a finite subset of* Ω*. Put* $\mathcal{B}_0 = \mathcal{B}$*. Let* $\mathcal{B}_i$ *and* $\mathcal{C}_i$ $(i = 1, 2, \ldots, m)$ *be subsets of* Ω *satisfying the following conditions.*

$$\mathrm{Inv}_{\mathcal{F}}(\mathcal{B}_{i-1}) = \mathcal{B}_i \cup \mathcal{C}_i, \tag{2.3}$$

$$\mathrm{o}(\mathcal{B}_i) \cap \mathcal{C}_i = \emptyset, \tag{2.4}$$

$$\mathrm{o}(\mathcal{B}_m) \subset \mathcal{B}. \tag{2.5}$$

Then $\mathcal{B}_m = \mathrm{Inv}_{\mathcal{F}}\Big(\mathrm{o}(\mathcal{B}_m) \cup \mathcal{C}_m\Big) \cap \mathrm{o}(\mathcal{B}_m)$.

Clearly, the condition which appears in the conclusion of the above theorem implies $\mathrm{Inv}_{\mathcal{F}}(\mathrm{o}(\mathcal{B}_m)) \subset \mathcal{B}_m$. Thus, $|\mathcal{B}_m|$ is an isolating neighbourhood for f by theorem 2.2.

The way of turning the above theorem into a constructive method, which will be discussed in detail in section 4, is to define the sets $\mathcal{B}_i$ and $\mathcal{C}_i$ inductively with respect to i, in such a way that the conditions (2.3) and (2.4) are satisfied until the condition (2.5) holds. Clearly, the decomposition (2.3) of $\mathrm{Inv}_{\mathcal{F}}(\mathcal{B}_{i-1})$ satisfying (2.4) is not unique and by varying it one can obtain a number of different isolating neighbourhoods.

The last of the theorems provides a formula for an index pair for the invariant part of a representable isolating neighbourhood constructed using the method given by the above theorem. Below, by an index pair we mean an index pair in the sense of [7], definition 5.1. This definition is slightly weaker that the one used in most references. We recall it in the theorem below.

THEOREM 2.4. *Let* $\mathcal{B}$ *be a finite subset of* Ω *such that*

$$\mathcal{B} = \mathrm{Inv}_{\mathcal{F}}\Big(\mathrm{o}(\mathcal{B}) \cup \mathcal{C}\Big) \cap \mathrm{o}(\mathcal{B})$$

for some $\mathcal{C} \subset \Omega$ *such that* $\mathcal{C} \cap \mathrm{o}(\mathcal{B}) = \emptyset$. *Then the pair*

$$Q = (Q_1, Q_0) = \Big(|(\mathrm{d}(\mathcal{B}) \cap \mathcal{F}(\mathcal{B})) \cup \mathcal{B}|,\ |\mathrm{d}(\mathcal{B}) \cap \mathcal{F}(\mathcal{B})|\Big)$$

satisfies the following conditions

$$f(Q_0) \cap Q_1 \subset Q_0, \tag{2.6}$$

$$f(Q_1 \setminus Q_0) \cap (Q_1 \setminus Q_0) \subset \mathrm{int}(Q_1), \tag{2.7}$$

$$\mathrm{Inv}_f(\mathrm{cl}(Q_1 \setminus Q_0)) \subset \mathrm{int}(Q_1 \setminus Q_0). \tag{2.8}$$

The induced map $f_Q : Q_1/Q_0 \to Q_1/Q_0$ *given by*

$$f_Q([x]) = \begin{cases} [f(x)] & \text{if } x, f(x) \in Q_1 \setminus Q_0 \\ [Q_0] & \text{otherwise} \end{cases} \tag{2.9}$$

is continuous, i.e. Q *is an index pair with respect to* f.

The Conley index theory for decompositions of isolated invariant sets defined in [9] is based on a different (more restrictive) definiton of an index pair than the one of J.W.Robbin and D.Salamon. In order to justify the usage of the algorithm described above for the computation of the Conley index for decompositions, we prove the following lemma. In the formulation and proof, the notation and terminology of [9] is used.

LEMMA 2.1. *Let* $Q = (Q_1, Q_0)$ *be a pair of compact subsets of a locally compact metric space* X *satisfying conditions (2.6),(2.7) and (2.8). Let* $\{D_b\}_{b\in B}$ *be a decomposition of* $\mathrm{cl}(Q_1 \setminus Q_0)$ *into a finite number of disjoint* *compact sets. Define:* $S = \mathrm{Inv}_f \mathrm{cl}(Q_1 \setminus Q_0)$, $A = 2^B$ *and* $S_b = D_b \cap S$ *for all* $b \in B$. *For each* $Z \in A$ *the continuous maps*

$$f^Z = f^Z_{(Q,\{D_b\})},\ p^Z = p^Z_{(Q,\{D_b\})} : Q_1/Q_0 \longrightarrow Q_1/Q_0$$

are defined by

$$p^Z([x]) = \begin{cases} [x] & \textit{if } x \in D_Z \\ [Q_0] & \textit{otherwise} \end{cases},$$

$$f^Z = f_Q \circ p^Z,$$

where f_Q *is given by (2.9) and* $D_Z = \bigcup_{b \in Z} D_b$. *Then,*

$$h(\{S_b\}, f, X) = \Big[Q_1/Q_0, \{f^Z\}_{Z \in A}\Big].$$

Recall that by $h(\{S_b\}, f, X)$ we denote the Conley index of the decomposition $\{S_b\}$ of S with respect to f.

3. Proofs

Proof of proposition 2.1. Let $x \in \mathrm{Inv}_f(|\mathcal{B}|)$. There exists a sequence $\{x_n\}_{n \in \mathbf{Z}} \subset |\mathcal{B}|$ such that $f(x_n) = x_{n+1}$ for all $n \in \mathbf{Z}$ and $x_0 = x$. Let $K_n \in \mathcal{B}$ be such that $x_n \in K_n$. By (2.2), $K_{n+1} \in \mathcal{F}(K_n)$. Hence $K_0 \in \mathrm{Inv}_{\mathcal{F}}(\mathcal{B})$ and therefore, $x = x_0 \in K_0 \subset |\mathrm{Inv}_{\mathcal{F}}(\mathcal{B})|$. □

Proof of theorem 2.1. Since $\{\mathcal{B}_n\}$ is a decreasing sequence of finite sets, there exists an $n_0 \in \mathbf{Z}^+$ such that $\mathcal{B}_n = \mathcal{B}_{n_0}$ for all $n \geq n_0$. Then, $\mathcal{B}_{n_0} = \mathcal{F}(\mathcal{B}_{n_0}) \cap \mathcal{B}_{n_0} \cap \mathcal{F}^{-1}(\mathcal{B}_{n_0})$. This means that for each $K \in \mathcal{B}_{n_0}$ there exist $J, L \in \mathcal{B}_{n_0}$ such that $K \in \mathcal{F}(J)$ and $L \in \mathcal{F}(K)$. It follows easily that $\mathrm{Inv}_{\mathcal{F}}(\mathcal{B}_{n_0}) = \mathcal{B}_{n_0}$. Since $\mathcal{B}_{n_0} \subset \mathcal{B}$, this proves that $\mathcal{B}_{n_0} \subset \mathrm{Inv}_{\mathcal{F}}(\mathcal{B})$. To show that the converse inclusion holds, notice that if $\{K_n\}_{n \in \mathbf{Z}} \subset \mathcal{B}$ satisfies $K_{n+1} \in \mathcal{F}(K_n)$ then it can be proved inductively on i that $\{K_n\} \subset \mathcal{B}_i$ for all $i \in \mathbf{Z}^+$. In particular, $K_0 \in \mathcal{B}_{n_0}$. □

Proof of theorem 2.2. Assume the contrary. There exists a double-sided sequence $\{x_n\} \subset |\mathcal{B}|$ such that $f(x_n) = x_{n+1}$ for all $n \in \mathbf{Z}$ and $x_0 \in \mathrm{bd}|\mathcal{B}|$. For each $n \in \mathbf{Z}$ chose $K_n \in \Omega$ such that $x_n \in K_n$. Clearly, K_0 can be chosen in such a way that $K_0 \notin \mathcal{B}$. Since $K_n \in \mathrm{o}(\mathcal{B})$ for all $n \in \mathbf{Z}$ and, by (2.2), $K_{n+1} \in \mathcal{F}(K_n)$, we obtain $K_0 \in \mathrm{Inv}_{\mathcal{F}}(\mathrm{o}(\mathcal{B})) = \mathcal{B}$, which is a contradiction. □

Proof of theorem 2.3. Let $\mathcal{U}_i = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \ldots \cup \mathcal{C}_i$, $i = 0, 1, \ldots, m$. We prove inductively that $\mathrm{Inv}_{\mathcal{F}}(\mathcal{B} \setminus \mathcal{U}_i) = \mathrm{Inv}_{\mathcal{F}}(\mathcal{B}_i)$. It is obvious for $i = 0$. Supposing that this equality holds for some $i < m$ and using (2.3) and (2.4), we obtain

$$\begin{aligned} \mathrm{Inv}_{\mathcal{F}}(\mathcal{B} \setminus \mathcal{U}_{i+1}) = \mathrm{Inv}_{\mathcal{F}}((\mathcal{B} \setminus \mathcal{U}_i) \setminus \mathcal{C}_{i+1}) \subset \mathrm{Inv}_{\mathcal{F}}(\mathcal{B} \setminus \mathcal{U}_i) \setminus \mathcal{C}_{i+1} = \\ = \mathrm{Inv}_{\mathcal{F}}(\mathcal{B}_i) \setminus \mathcal{C}_{i+1} = \mathcal{B}_{i+1}. \end{aligned}$$

Hence $\mathrm{Inv}_{\mathcal{F}}(\mathcal{B} \setminus \mathcal{U}_{i+1}) \subset \mathrm{Inv}_{\mathcal{F}}(\mathcal{B}_{i+1})$. Since $\mathcal{B}_{i+1} \subset \mathcal{B} \setminus \mathcal{U}_{i+1}$ by (2.4), the converse inclusion holds and the proof is finished. By (2.4) and (2.5), $\mathrm{o}(\mathcal{B}_m) \subset \mathcal{B} \setminus \mathcal{U}_m$. It follows that

$$\begin{aligned} \mathrm{Inv}_{\mathcal{F}}\Big(\mathrm{o}(\mathcal{B}_m) \cup \mathcal{C}_m\Big) \cap \mathrm{o}(\mathcal{B}_m) \subset \mathrm{Inv}_{\mathcal{F}}\Big((\mathcal{B} \setminus \mathcal{U}_m) \cup \mathcal{C}_m\Big) \cap \mathrm{o}(\mathcal{B}_m) \subset \\ \subset \mathrm{Inv}_{\mathcal{F}}(\mathcal{B} \setminus \mathcal{U}_{m-1}) \cap \mathrm{o}(\mathcal{B}_m) = \mathrm{Inv}_{\mathcal{F}}(\mathcal{B}_{m-1}) \cap \mathrm{o}(\mathcal{B}_m) = \\ = (\mathcal{B}_m \cup \mathcal{C}_m) \cap \mathrm{o}(\mathcal{B}_m) = \mathcal{B}_m. \end{aligned}$$

Since the reverse inclusion is obvious, the proof is finished. □

Proof of theorem 2.4. Notice that $\mathrm{cl}(Q_1 \setminus Q_0) = |\mathcal{B}|$. Therefore, (2.8) follows immediately from theorem 2.2. Moreover, by (2.1),

$$\begin{aligned} &f(Q_1 \setminus Q_0) \cap (Q_1 \setminus Q_0) \subset |\mathcal{B}| \cap \mathrm{int}|\mathcal{F}(\mathcal{B})| \subset \\ &\quad \subset \mathrm{int}|\mathcal{F}(\mathcal{B})| \cap \mathrm{int}|\mathrm{o}(\mathcal{B})| \subset \mathrm{int}|\mathcal{F}(\mathcal{B}) \cap \mathrm{o}(\mathcal{B})| \subset \mathrm{int}(Q_1), \end{aligned}$$

so that (2.7) holds.

We prove (2.6) by contradiction. Assume that there exists an $x \in Q_0$ such that $f(x) \in Q_1 \setminus Q_0$. Let $\bar{M} \in \mathrm{d}(\mathcal{B}) \cap \mathcal{F}(\mathcal{B})$ and $L \in \mathcal{B}$ satisfy the following condition

$$x \in \bar{M} \text{ and } f(x) \in L.$$

Let $M \in \mathcal{B}$ be such that $\bar{M} \in \mathcal{F}(M)$. By assumption,

$$\mathcal{B} = \mathrm{Inv}_{\mathcal{F}}\Big(\mathrm{o}(\mathcal{B}) \cup \mathcal{C}\Big) \cap \mathrm{o}(\mathcal{B}).$$

Hence there exist double-sided sequences $\{M_n\}, \{L_n\} \subset \mathrm{o}(\mathcal{B}) \cup \mathcal{C}$ such that $M_0 = M, L_0 = L$ and, for each $n \in \mathbf{Z}$, $M_{n+1} \in \mathcal{F}(M_n)$ and $L_{n+1} \in \mathcal{F}(L_n)$. For $n \in \mathbf{Z}$ put:

$$K_n = \begin{cases} M_{n+1} & \text{if } n \leq -1 \\ \bar{M} & \text{if } n = 0 \\ L_{n-1} & \text{if } n \geq 1 \end{cases}.$$

Then, for all all $n \in \mathbf{Z}$, $K_{n+1} \in \mathcal{F}(K_n)$. It follows that $\bar{M} = K_0 \in \mathrm{Inv}_{\mathcal{F}}(\mathrm{o}(\mathcal{B}) \cup \mathcal{C}) \cap \mathrm{o}(\mathcal{B}) = \mathcal{B}$, which is a contradiction.

To finish the proof, notice that the conditions (2.6),(2.7) and (2.8) imply the assumptions of theorem 4.3 in [7]. □

Proof of lemma 2.1. Put $\bar{Q}_1 = Q_1$ and

$$\bar{Q}_0 = \Big\{x \in Q_1 : f(x) \notin Q_1 \setminus Q_0\Big\} = Q_1 \cap f^{-1}\Big(Q_0 \cup (X \setminus Q_1)\Big).$$

One can easily see that $f(\bar{Q}_0) \cap \bar{Q}_1 \subset \bar{Q}_0$, $f(\bar{Q}_1 \setminus \bar{Q}_0) \subset \bar{Q}_1$ and $S = \mathrm{Inv}_f \mathrm{cl}(\bar{Q}_1 \setminus \bar{Q}_0) \subset \mathrm{int}(\bar{Q}_1 \setminus \bar{Q}_0)$. By (2.6) and (2.7),

$$\bar{Q}_0 = f^{-1}\Big(Q_0 \cup \mathrm{cl}(X \setminus Q_1)\Big) \cap Q_1,$$

so that $\bar{Q}_0$ is compact. Moreover, $\{\bar{D}_b\}_{b\in B} = \{D_b \cap \mathrm{cl}(\bar{Q}_1 \setminus \bar{Q}_0)\}_{b\in B}$ is a decomposition of $\mathrm{cl}(\bar{Q}_1 \setminus \bar{Q}_0)$ such that $\bar{D}_b \cap S = S_b$. This means that $\bar{Q} = (\bar{Q}_1, \bar{Q}_0)$ is an index pair for S compatibile with the decomposition $\{S_b\}$ of S in the sense of [9], definition 3.1. Therefore,

$$h(\{S_b\}, f, X) = \Big[\bar{Q}_1/\bar{Q}_0, \{\bar{f}^Z\}_{Z\in A}\Big],$$

where $\bar{f}^Z = f^Z_{(\bar{Q},\{\bar{D}_b\})}$. Our task is to prove that the objects $\Big(\bar{Q}_1/\bar{Q}_0, \{\bar{f}^Z\}_{Z\in A}\Big)$ and $\Big(Q_1/Q_0, \{f^Z\}_{Z\in A}\Big)$ are isomorphic in the $\mathcal{H}top_{[A]}$ category. To see this, notice that the inclusion-induced map $i : Q_1/Q_0 \to \bar{Q}_1/\bar{Q}_0$ induces the morphism

$$[i,0] : \Big(Q_1/Q_0, \{f^Z\}_{Z\in A}\Big) \longrightarrow \Big(\bar{Q}_1/\bar{Q}_0, \{\bar{f}^Z\}_{Z\in A}\Big),$$

whose inverse is the morphism

$$\Big[\{g^Z\}_{Z\in A}, 1\Big] : \Big(\bar{Q}_1/\bar{Q}_0, \{\bar{f}^Z\}_{Z\in A}\Big) \longrightarrow \Big(Q_1/Q_0, \{f^Z\}_{Z\in A}\Big),$$

where $g^Z = f_{(\bar{Q},Q)} \circ r^Z_{(\bar{Q},\{\bar{D}_b)}$ for all $Z \in A$ and the map $f_{(\bar{Q},Q)} : \bar{Q}_1/\bar{Q}_0 \to Q_1/Q_0$ is induced by f. $\Box$

4. Algorithms

This section contains the discussion of algorithms based on theorems 2.1-2.4. Below $s \in \mathbf{N}$ and $d_1, d_2, \ldots, d_s \in \mathbf{R}^+$ are fixed and have the same meaning as in sections 2 and 3. In order to represent elements of Ω we use the ELEM type. We shall not go into the details of the definition of this type. If E is a variable of type ELEM then by E we shall also denote the member of Ω represented by the value of E. The type **set of** ELEM will be used to represent finite subsets of Ω. For a variable A of that type, by A we shall also denote the set represented by the value of A.

We start with the algorithm for finding $\mathrm{Inv}_{\mathcal{F}}$ based on theorem 2.1.

ALGORITHM 4.1.

```
function INV ( A: set of ELEM ) : set of ELEM;
var
   I,U,B,C          : set of ELEM;
   E                : ELEM;
   CHANGED,EXIT     : BOOLEAN;
begin
   B:=A;
   repeat
      U:=$\emptyset$; C:=B; CHANGED:= false;
      for E $\in$ B do
      begin
         I:= $\mathcal{F}$(E);
         if I $\cap$ C = $\emptyset$ then
            begin
             C:=C $\setminus$ {E};
             CHANGED:= true;
            end
         else
            U:=U $\cup$ I;
      end;
      CHANGED:=CHANGED or not C $\subset$ U;
      B:=C $\cap$ U;
   until not CHANGED;
   return B;
end;
```

Theorem 4.1. *The* Inv *function called with its argument* A *representing a finite set* $\mathcal{A} \subset \Omega$ *returns* $\mathrm{Inv}_{\mathcal{F}}(\mathcal{A})$.

The above theorem follows immediately from theorem 2.1 and the following lemma.

Lemma 4.1. *For a given finite nonempty set* $\mathcal{B} \subset \Omega$ *let* $\mathrm{E}(\mathcal{B})$ *and* $\mathrm{c}(\mathcal{B})$ *denote the values of the* B *and* changed *variables after going through the body of the repeat-until loop in algorithm 4.1, with the initial value of* B *set to* $\mathcal{B}$. *Then,*

$$\mathrm{Inv}_{\mathcal{F}}(\mathcal{B}) \subset \mathrm{E}(\mathcal{B}) \subset \mathcal{B} \cap \mathcal{F}(\mathcal{B}) \cap \mathcal{F}^{-1}(\mathcal{B})$$

and

$$\mathrm{c}(\mathcal{B}) = \begin{cases} \textbf{true} & \textit{if } \mathcal{B} \neq \mathrm{E}(\mathcal{B}) \\ \textbf{false} & \textit{otherwise} \end{cases} . \tag{4.1}$$

Proof. Fix $\mathcal{B} \subset \Omega$ satisfying the required conditions and suppose that the for-loop is repeated exactly m times while computing $\mathrm{E}(\mathcal{B})$ and $\mathrm{c}(\mathcal{B})$. Let e_n, $\mathcal{C}_n$, $\mathcal{U}_n$ and c_n denote the values of the e, C, U and changed variables immediately before performing the first instruction in the loop for the $(n+1)$-th time, $n = 0, 1, \ldots, m-1$. By $\mathcal{C}_m, \mathcal{U}_m$ and c_m denote the values of the corresponding variables immediately after exiting the loop. The following relations hold for $n = 0, 1, \ldots, m-1$.

$$\mathcal{C}_{n+1} = \begin{cases} \mathcal{C}_n \setminus \{e_n\} & \text{if } \mathcal{F}(e_n) \cap \mathcal{C}_n = \emptyset \\ \mathcal{C}_n & \text{otherwise} \end{cases} , \tag{4.2}$$

$$\mathcal{U}_{n+1} = \begin{cases} \mathcal{U}_n \cup \mathcal{F}(e_n) & \text{if } \mathcal{F}(e_n) \cap \mathcal{C}_n \neq \emptyset \\ \mathcal{U}_n & \text{otherwise} \end{cases} , \tag{4.3}$$

$$c_{n+1} = c_n \vee \mathcal{C}_{n+1} \neq \mathcal{C}_n, \tag{4.4}$$

$$\mathcal{C}_0 = \mathcal{B} = \{e_0, e_1, \ldots, e_{m-1}\} \tag{4.5}$$

It follows from (4.2) that $\mathrm{Inv}_{\mathcal{F}}(\mathcal{C}_{n+1}) = \mathrm{Inv}_{\mathcal{F}}(\mathcal{C}_n)$ for $n = 0, 1, \ldots m-1$. Hence,

$$\mathrm{Inv}_{\mathcal{F}}(\mathcal{B}) = \mathrm{Inv}_{\mathcal{F}}(\mathcal{C}_0) = \mathrm{Inv}_{\mathcal{F}}(\mathcal{C}_m) \subset \mathcal{C}_m. \tag{4.6}$$

If $b \in \mathrm{Inv}_{\mathcal{F}}(\mathcal{C}_m)$ then $\mathcal{F}(b) \cap \mathcal{C}_m \neq \emptyset$ and therefore $\mathcal{F}(b) \cap \mathcal{C}_n \neq \emptyset$ for all $n \in \{0, 1, \ldots, m\}$. Hence by (4.3), (4.5) and (4.6),

$$\mathrm{Inv}_{\mathcal{F}}(\mathcal{B}) \subset \mathcal{F}(\mathrm{Inv}_{\mathcal{F}}(\mathcal{C}_m)) \subset \mathcal{U}_m.$$

By (4.6),

$$\mathrm{Inv}_{\mathcal{F}}(\mathcal{B}) = \mathrm{Inv}_{\mathcal{F}}(\mathcal{C}_m) \subset \mathcal{U}_m \cap \mathcal{C}_m = \mathrm{E}(\mathcal{B}).$$

Let us prove that $\mathrm{E}(\mathcal{B}) \subset \mathcal{B} \cap \mathcal{F}(\mathcal{B}) \cap \mathcal{F}^{-1}(\mathcal{B})$ now. By (4.5),

$$\mathcal{U}_m \subset \mathcal{F}(\{e_0, e_1, \ldots, e_n\}) \subset \mathcal{F}(\mathcal{B}). \tag{4.7}$$

Since $\mathcal{C}_n \subset \mathcal{B}$ for $n = 0, 1, \ldots, m$, $e_n \notin \mathcal{C}_m$ if $\mathcal{F}(e_n) \cap \mathcal{B} = \emptyset$. By (4.5) and $\mathcal{C}_m \subset \mathcal{B}$, $\mathcal{C}_m \subset \mathcal{B} \cap \mathcal{F}^{-1}(\mathcal{B})$. By (4.7),

$$\mathrm{E}(\mathcal{B}) = \mathcal{C}_m \cap \mathcal{U}_m \subset \mathcal{B} \cap \mathcal{F}(\mathcal{B}) \cap \mathcal{F}^{-1}(\mathcal{B}).$$

It remains to prove (4.1). Since $c_0 =$ **false** and $\mathcal{C}_{n+1} \subset \mathcal{C}_n$, an easy inductive reasoning based on (4.4) shows that

$$c_m = \begin{cases} \textbf{true} & \text{if } \mathcal{C}_0 \neq \mathcal{C}_m \\ \textbf{false} & \text{otherwise} \end{cases}.$$

Since $\mathrm{E}(\mathcal{B}) = \mathcal{C}_m \cap \mathcal{U}_m$, $\mathrm{c}(\mathcal{B}) = c_m \vee \mathcal{C}_m \not\subset \mathcal{U}_m$ and $\mathcal{C}_0 = \mathcal{B}$, (4.1) follows. □

Now we discuss the algorithm for finding an index pair contained in a given representable set. Its idea is contained in theorems 2.2, 2.3 and 2.4. In order to represent pairs of subsets of $\mathbf{R}^s$ we use the following structure.

```
Pair = record
          Q1,Q0 : set of Elem;
       end;
```

The pair represented by a variable Q of type Pair is given by

$$\Big(|\mathrm{Q.Q1}|, |\mathrm{Q.Q0}|\Big).$$

In what follows, we shall not discuss the methods of computing $\mathrm{d}(\mathcal{A})$,$\mathrm{o}(\mathcal{A})$ and $\mathcal{F}(\mathcal{A})$ for a given set $\mathcal{A} \subset \Omega$, since they are quite obvious.

Algorithm 4.2.

```
function IndexPair ( A : set of Elem ) : Pair;
var
   D,B,C  : set of Elem;
   P      : Pair;
begin
   B:=A; D:=$d(\Omega \setminus B)$;
   repeat
      B:=Inv(B);
      Decompose(B,D,B,C);
   until Found(D,B,C);
   P.Q1:=$(d(B) \cap \mathcal{F}(B)) \cup B$;
   P.Q0:=$d(B) \cap \mathcal{F}(B)$;
   return P;
end;
```

Theorem 4.2. *Assume that the procedure*

procedure Decompose (A,D : **set of** Elem; **var** B,C : **set of** Elem);

and the function

```
function Found ( D,B,C : set of Elem ) : Boolean;
```

are defined in such a way that the following conditions are satisfied.
1°. *After calling* Decompose(A,D,B,C)*,*

$$\mathrm{o(B)} \cap \mathrm{C} = \emptyset,\ \mathrm{B} \cup \mathrm{C} = \mathrm{A} \quad \textit{and} \quad \Big(\mathrm{A} \cap \mathrm{D} \neq \emptyset \Rightarrow \mathrm{C} \neq \emptyset.\Big)$$

2°. *The following two implications hold:*

$$\mathrm{B} \cap \mathrm{D} \neq \emptyset \Rightarrow \textsc{Found}(\mathrm{D,B,C}) = \mathbf{false}\ ,$$

$$\mathrm{B} \cap \mathrm{D} = \emptyset \quad \textit{and} \quad \mathrm{C} = \emptyset \Rightarrow \textsc{Found}(\mathrm{D,B,C}) = \mathbf{true}\ .$$

Then, the value returned by the function IndexPair*, called with its argument* A *representing a finite subset* $\mathcal{A}$ *of* Ω*, represents an index pair for* f*.*

Proof. We begin with showing that the algorithm stops. Assume the contrary. Let $\mathcal{D} = \mathrm{d}(\Omega \setminus \mathcal{A})$. By $\mathcal{B}_n$ and $\mathcal{C}_n$ we denote the value of the B and C variables immediately before performing the first instruction in the repeat-until loop for the $(n+1)$-th time. Since $\mathcal{B}_{n+1} \subset \mathcal{B}_n$, there exists an $m \in \mathbf{Z}^+$ such that $\mathcal{B}_{m+1} = \mathcal{B}_m$. Then, $\mathcal{C}_{m+1} = \emptyset$. By 1°, $\mathcal{B}_m \cap \mathcal{D} = \emptyset$ so that by 2°, Found($\mathcal{D}$,$\mathcal{B}_{m+1}$,$\mathcal{C}_{m+1}$) = **true** which means that the loop is exited after a finite time.

Suppose that the loop in the IndexPair function is repeated exactly m times after the function is called with the A argument representing a finite set $\mathcal{A} \subset \Omega$. Then, the D variable represents the set $\mathcal{D} = \mathrm{d}(\Omega \setminus \mathcal{A})$. As before, let $\mathcal{B}_n$ and $\mathcal{C}_n$ denote the consecutive values of the B and C variables immediately before performing the B:=Inv(B) instruction for the $(n+1)$-th time. $\mathcal{B}_m$ and $\mathcal{C}_m$ denote the values assumed by these variables on exiting the loop. By 1°,

$$\mathrm{Inv}_{\mathcal{F}}(\mathcal{B}_{n-1}) = \mathcal{B}_n \cup \mathcal{C}_n \text{ and } \mathrm{o}(\mathcal{B}_n) \cap \mathcal{C}_n = \emptyset$$

for $n = 1, 2, \ldots, m$. Since Found($\mathcal{D}$,$\mathcal{B}_m$,$\mathcal{C}_m$)=**true**, 2° implies $\mathcal{B}_m \cap \mathcal{D} = \emptyset$. Since $\mathcal{B}_{n+1} \subset \mathcal{B}_n \subset \mathcal{A}$ for $n = 0, 1, \ldots, m-1$ and $\mathcal{D} = d(\Omega \setminus \mathcal{A})$, it follows that $\mathrm{o}(\mathcal{B}_m) \subset \mathcal{A}$. In order to finish the proof, use theorems 2.3 and 2.4. □

The following examples show two ways of defining the Found function and the Decompose procedure satisfying the assumptions of the above lemma. In what follows, for a variable A of type **set of** Elem by < A > we denote the subset of $\mathbf{R}^s$ represented by the value of A.

Example 4.1. Define the procedure Decompose in such a way that after calling Decompose(A,D,B,C) for A representing a finite subset of Ω, < C > is the union of all components of < A > containing an element of the subset of Ω represented by D and < B >=< A > \ < C >. Set

$$\textsc{Found}(\mathrm{D,B,C}) = \begin{cases} \mathbf{true} & \text{if } \mathrm{D} \cap \mathrm{B} = \emptyset \\ \mathbf{false} & \text{otherwise} \end{cases}\ .$$

Note that in this case, the loop in the IndexPair function is run through only once. Thus the amount of computations is often considerably lower than in the case of different choices of FOUND and DECOMPOSE (cf example 4.2 below). The disadvantage of this particular choice is that the isolated invariant set detected using the algorithms described above may be relatively small (roughly speaking, smaller values of the C variable in the INDEXPAIR function give a better chance of finding a bigger and hence more interesting isolated invariant set).

EXAMPLE 4.2. Introduce a linear order in Ω and define DECOMPOSE in such a way that after performing the instruction DECOMPOSE(A,D,B,C), $<\mathrm{C}>$ is the component of $<\mathrm{A}>$ which contains the minimal element of the intersection of the subsets of $\mathcal{A}$ represented by A and D. If the intersection is empty, set C to be empty. The value given to B is such that the condition 1° in theorem 4.2 holds (i.e. $<\mathrm{B}>=<\mathrm{A}>\setminus<\mathrm{C}>$). The FOUND function is defined by

$$\textsc{Found}(\mathrm{D,B,C}) = \begin{cases} \textbf{true} & \text{if } \mathrm{C} = \mathrm{D} \cap \mathrm{B} = \emptyset \\ \textbf{false} & \text{otherwise} \end{cases}.$$

5. Periodic points in the Henon attractor

The Henon map $f : \mathbf{R}^2 \to \mathbf{R}^2$, introduced by M.Henon in [1] is given by the formula

$$f(x,y) = (1 - ax^2 + y, bx)$$

where $a = 1.4$ and $b = 0.3$. Its remarkable property is that it maps the tetragon G whose vertices are $(-1.33, 0.42)$, $(1.32, 0.133)$, $(1.245, -0.14)$ and $(-1.06, -0.5)$ into its interior. Therefore, the set

$$A = \bigcap_{n=1}^{\infty} f^n(G)$$

is an attractor of f, which will be called the Henon attractor. One can easily show that G (and hence also A) contains exactly one stationary point whose coordinates will be denoted by (x_0, y_0). Numerical experiments carried out by Henon revealed that the structure of the attractor is very complicated and the trajectories of points in this set wander around it in a chaotic manner. Recently P.Zgliczyński (see [10]) gave a computer-assisted proof of the existence of an invariant set of f^7 contained in A such that the restriction of f^7 to this set is semiconjugate with the shift map on two symbols. The semiconjugacy obtained by that method is such that each periodic sequence is an image of a periodic sequence of f^7 with the same principal period. In particular, this means that there are periodic points of f^7 of arbitrary period in A. Below we shall find the set of periods (with respect to f) of all periodic orbits in A. The main tool we are going to use is the Conley index for decompositions of isolated invariant sets, computed using the algorithms described in the previous section.

We start with a lemma concerning the existence of local Lyapunov functions for the Henon map. In what follows, $U = \{(x,y) \in \mathbf{R}^2 : x \geq 1/2\}$.

LEMMA 5.1. *The function* $V : \mathbf{R}^2 \to \mathbf{R}$ *given by*

$$\begin{aligned} V(x,y) = & -(x-x_0)^2 + (x+y-x_0-y_0)^2 = \\ = & (y-y_0)^2 + 2(x-x_0)(y-y_0) \end{aligned}$$

satisfies the following condition for each $(x,y)\in\mathbf{R}^2$ *:*

$$\text{if } x\geq 1/2 \text{ then } V(f(x,y))\leq V(x,y)$$

The inequality is strict if $(x,y)\neq(x_0,y_0)$.

Proof. Let $g(x,y)=V(f(x,y))-V(x,y)$. Since (x_0,y_0) is a fixed point of f,

$$\begin{aligned} g(x,y) = {} & (bx-y_0)^2+2(1-ax^2+y-x_0)(bx-y_0) \\ & -(y-y_0)^2-2(x-x_0)(y-y_0) = \\ = {} & b^2(x-x_0)^2-2ab(x-x_0)^2(x+x_0)+ \\ & +(2b-2)(y-y_0)(x-x_0)-(y-y_0)^2. \end{aligned}$$

Thus, $Dg(x_0,y_0)=0$ and, for $x\geq 1/2$, the matrix

$$\begin{bmatrix} \frac{\partial^2 g}{\partial x^2} & \frac{\partial^2 g}{\partial x\partial y} \\ \frac{\partial^2 g}{\partial x\partial y} & \frac{\partial^2 g}{\partial y^2} \end{bmatrix}_{(x,y)} = \begin{bmatrix} 2b^2-4ab(3x-x_0) & 2-2b \\ 2-2b & -2 \end{bmatrix}$$

is negative definite, since as one can easily check,

$$x_0=\frac{-0.7+\sqrt{6.09}}{2.8}<0.7,$$

so that $2b^2-4ab(3x-x_0)<0$ and $-4b^2+8ab(3x-x_0)-(2-2b)^2>0$. By the Taylor's formula, for all $(x,y)\in U\setminus\{(x_0,y_0)\}$ there exists $\theta\in U$ such that

$$g(x,y)=\frac{1}{2}D^2g(\theta)(x-x_0,y-y_0)^2.$$

Since the right-hand side is negative provided $(x,y)\neq(x_0,y_0)$, the proof is finished. □

Corollary 5.1. *The only bounded trajectory of* f *contained in* U *is the stationary trajectory through* (x_0,y_0).

Now, let us describe the results of a computer-assisted computation of the Conley index of a certain decomposition of an isolated invariant set S contained in

$$P=[-85/69,85/69]\times[-86/230,86/230]\setminus[-85/69,10/69)\times[-86/230,0).$$

In order to find its isolating neighbourhood and index pair, we applied the procedure described in the previous sections with $s=2$, $d_1=1/69$ and $d_2=1/230$. The Found and Decompose functions we used are discussed in Example 4.2 (with any linear ordering of Ω - there is only one connected component of the subset of R^s represented by $\text{Inv}_{\mathcal{F}}\{K\in\Omega : K\subset P\}$). The function $\mathcal{F}:\Omega\to 2^\Omega$ was defined in such a way that for each $K\in\Omega$, $\mathcal{F}(K)$ is the smallest representable rectangle containing $f(K)$ in its interior. Clearly, the computation of $\mathcal{F}$ can be performed in an efficient and straighforward manner using only integer numbers, so that there are no rounding errors.

The application of the algorithms results in an index pair $Q=(Q_1,Q_0)$ with $\text{cl}(Q_1,Q_0)$ being a disjoint union of a number of compact sets. We chose a decomposition of the isolating neighbourhood into 5 compact sets, in this way fixing a decomposition of S into 5 disjoint subsets, which we shall denote by S_b, $b \in B = \{1, 2, 3, 4, 5\}$. Put $A = 2^B$. An index pair compatibile with the decomposition we have chosen was found using the formula in theorem 4. In order to compute the endomorphism induced by the index map on cohomology, we found a strong deformation retraction of the quotient space of the index pair to a graph. Then, we computed its spanning tree, obtaining a basis for the fundamental group of the quotient space (see [8]). Then, we examined the homotopy types of the images of the fundamental cycles under f in order to obtain the endomorphism induced on one-dimensional cohomology. For a detailed description of the algorithm the reader is referred to [5]. The total computation time (on a PC computer with INTEL 486DX 66MHz processor) was about 2 minutes. The result of the computation follows.

$$h^q(\{S_b\}, f, \mathbf{R}^2) = 0 \quad \text{for } q \neq 1,$$
$$h^1(\{S_b\}, f, \mathbf{R}^2) = \left[V, \{\varphi^Z\}_{Z \in A}\right],$$

where $V = \mathbf{Q}^{18}$, $\varphi^Z = p^Z \circ \varphi$, φ is the linear endomorphism of V having the following matrix in the standard basis $(e_i)_{i=1}^{18}$ of V:

$$\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0
\end{bmatrix}$$

and $p^Z : V \to V$ is the projection onto the subspace $\bigoplus_{i \in Z} V_i \subset V$ being zero on $\bigoplus_{i \in B \setminus Z} V_i$, where

$$V_i = \begin{cases} span\{e_1\} & \text{if } i = 1, \\ span\{e_2, e_3, e_4, e_5\} & \text{if } i = 2, \\ span\{e_6, e_7\} & \text{if } i = 3, \\ span\{e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}\} & \text{if } i = 4, \\ span\{e_{14}, e_{15}, e_{16}, e_{17}, e_{18}\} & \text{if } i = 5. \end{cases}$$

In what follows, for $b \in B$ instead of writing $\varphi^{\{b\}}$ we shall write briefly φ^b. Finally, we checked using the procedure given by theorems 2.1 and 4.1

that (for $d_1 = 1/138$ and $d_2 = 1/460$ this time) the set $\mathcal{C} = \mathrm{Inv}_{\mathcal{F}}\Big(\{K \in \Omega : K \subset P\}\Big)$ consists of rectangles contained in G. Hence by proposition 2.1, $\mathrm{Inv}_f(P) \subset |\mathcal{C}| \subset G$. Hence $\mathrm{Inv}_f(P) \subset \mathrm{Inv}_f(G) = A$. Clearly, this proves that $S \subset A$. The main result of this section follows.

THEOREM 5.1. *The Henon attractor contains periodic orbits of all principal periods except for 3 and 5.*

Proof. In the proof, we shall use the properties of the Conley index for decompositions of isolated invariant sets. Define the continuous map

$$p : S \longrightarrow \Pi = \prod_{n \in \mathbf{Z}^+} B$$

by

$$p(x) = (b_i)_{i=0}^{\infty} \;\Leftrightarrow\; \forall_{i \in \mathbf{Z}^+}\, f^i(x) \in S_{b_i}.$$

It follows from theorem 4.5 in [9] that, if a sequence $\bar{b} = (b_i)_{i=0}^{\infty}$ is of principal period T and

$$\mathrm{tr}\Big(\varphi^{b_0} \circ \varphi^{b_1} \circ \ldots \circ \varphi^{b_{T-1}}\Big) \neq 0$$

then $\bar{b}$ is an image (under p) of a periodic point of f with the same principal period (notice that, in the terminology of [9], the left-hand side is equal, up to a sign, to the Lefschetz number of $\mathcal{P}_{(b_{T-1}, b_{T-2}, \ldots, b_0)}(h^*(\{S_b\}, f, \mathbf{R}^2)))$. This property allows to produce lower bounds for the number of periodic orbits of f of a fixed principal period.

In order to prove that there are periodic orbits of principal periods 1,2 and 4, notice that

$$\mathrm{tr}(\varphi^1) = tr(\varphi^2 \circ \varphi^4) = \mathrm{tr}(\varphi^3 \circ \varphi^4 \circ \varphi^2 \circ \varphi^5) = -1.$$

To show the existence of periodic trajectories of any principal period $T \geq 6$ notice that the composition $\bar{\varphi} = \varphi^5 \circ \varphi^5 \circ \varphi^3 \circ \varphi^4 \circ \varphi^2 \circ \varphi^5$ has the following properties.

$$\mathrm{im}\ \bar{\varphi} = \mathrm{span}\{e_{18}\},$$

$$\bar{\varphi}(e_1) = \bar{\varphi}(e_{18}) = -e_{18}.$$

Thus,

$$\mathrm{tr}\Big((\varphi^1)^{T-6} \circ \bar{\varphi})\Big) = (-1)^{T+1}$$

and therefore there exists a periodic orbit of principal period T. It remains to prove that there are no periodic trajectories of principal period 3 and 5. The cruicial observation in this part of the proof is that the set of all T-periodic points of f in A is contained in $|\mathcal{R}_T|$, where

$$\mathcal{R}_T = \mathrm{Inv}_{\mathcal{F}}\Big\{K \in \Omega : K \cap G \neq \emptyset \text{ and } K \in \mathcal{F}^T(K)\Big\}.$$

Using the algorithm given by theorems 2.1 and 4.1, it can be checked with the aid of a computer that, for T=3 and 5, all rectangles in $\mathcal{R}_T$ are contained in U. Hence, by corollary 1, the stationary point (x_0, y_0) is the only periodic point of period 3 or 5 in A and therefore, A contains no periodic orbits of principal period 3 and 5. This finishes the proof of theorem 5.1. □

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