The Conley Index and Symbolic Dynamics

Andrzej Szymczak

Abstract

Suppose that N is an isolating neighborhood for S with respect to a continuous map f which decomposes into a disjoint union of finitely many compact sets. We establish a relationship between the Conley indices of subsets of S which appear in a natural way when studying the behavior of f from the viewpoint of symbolic dynamics. The relationship is then used to prove a simple criterion for chaos in an isolated invariant set.

1991 Mathematics Subject Classification : 54H20, 34C35

0. *Introduction*

The Conley index has been an important tool in the study of the qualitative properties of dynamical systems. Recently K.Mischaikow and M.Mrozek [4] observed that it can be applied for the study of chaotic behavior of a dynamical system on an isolated invariant set. The assumptions of the criterion for chaos proved there were successfully checked in the case of the Lorenz equations by means of numerical computations [3]. The aim of this paper is to provide another method of detecting chaos based on the Conley index.

Let S be an isolated invariant set with respect to a map $f : X \to X$, which admits an isolating neighborhood N decomposing into a disjoint union of compact sets N_1, N_2, $\ldots N_k$. The fundamental question one encounters when dealing with the dynamics of f on S is whether for a given sequence $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_{l-1})$ of members of $\{1, 2, \ldots, k\}$ there exists an $x \in S$ such that $f^i(x) \in N_{\sigma_{(i \ mod \ l)}}$ for each nonnegative integer i. If the answer is affirmative for sufficiently many sequences σ then there exists a semiconjugacy defined on S onto the phase space of the one-sided shift on k symbols, which can be treated as a lower bound for the complexity of the dynamics of f on S. Our approach to this question is based on the observation that the answer to the question is affirmative if and only if

$$Inv_{f^l} \bigcap_{i=0}^{l-1} f^{-i}(N_{\sigma_i}) \neq \emptyset.$$

Since the set on the left-hand side is an isolated invariant set with respect to f^l, its Conley index is defined. Furthermore, if the index is nontrivial then the set is nonempty. We conclude that for the study of the dynamical behavior of f on S, it is important to know the relationship between those Conley indices for different sequences σ. We shall give such a relationship and apply it to prove a simple criterion for chaos.

1. *Preliminaries*

$\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Z}^+$ and $\mathbf{Q}$ will stand for the sets of natural, integer, nonnegative integer and rational numbers, respectively. For a pair (Q_1, Q_0) of compact subsets of a topological space by Q_1/Q_0 we shall denote the topological space resulting from Q_1 when the points of Q_0 are identified to a single point denoted by $[Q_0]$. If X is a metric space then there exists a metric on the quotient space Q_1/Q_0 which is compatibile with the quotient topology. Therefore, quotient spaces of this form will be treated as metric spaces.

$\mathcal{V}$ and $\mathcal{V}_G$ will denote the categories of vector spaces and graded vector spaces over a fixed field F. $\mathcal{H}top$ will stand for the category of pointed topological spaces whose morphisms are homotopy classes of basepoint preserving continuous maps. In order to simplify the notation, for a map g of pointed topological spaces by g we shall also denote its homotopy class. By H^* we mean a cohomology functor with coefficients in F. The range category of H^* is $\mathcal{V}_G$. Instead of writing $H^*(g)$ we shall often write briefly g^*. We shall make use of the fixed point index for Euclidean Neighborhood Retracts (ENR's) as defined in [1].

Now we proceed to the definition of the category of objects equipped with a morphism, which was used in [13] to define the Conley index. Let $\mathcal{K}$ be a category. We define the category of objects equipped with a morphism over $\mathcal{K}$, denoted by $\mathcal{K}_m$ as follows. Let

$$Ob(\mathcal{K}_m) = \{(W, \varphi) : W \in Ob(\mathcal{K}) \quad and \quad \varphi \in Mor_{\mathcal{K}}(W, W)\}.$$

For $(W, \varphi), (W', \varphi') \in Ob(\mathcal{K}_m)$ we put

$$Mor_{\mathcal{K}_m}((W, \varphi), (W', \varphi')) = M((W, \varphi), (W', \varphi'))/ \equiv$$

where

$$M((W, \varphi), (W', \varphi')) = \{\psi \in Mor_{\mathcal{K}}(W, W') : \psi \circ \varphi = \varphi' \circ \psi\} \times \mathbf{Z}^+$$

and $\equiv$ is the equivalence relation in the above set defined by

$$(\psi_1, n_1) \equiv (\psi_2, n_2) \iff \exists_{k \in \mathbf{Z}^+} \quad \psi_2 \circ \varphi^{n_1 + k} = \psi_1 \circ \varphi^{n_2 + k}.$$

The morphism represented by $(\psi, n) \in M((W, \varphi), (W', \varphi'))$ will be denoted by $[\psi, n]$. We note that, by the definition of $\equiv$, for any $m \in \mathbf{Z}^+$,

$$[\psi, n] = [\psi \circ \varphi^m, n + m] = [(\varphi')^m \circ \psi, n + m].$$

The composition of morphisms $[\psi, n] \in Mor_{\mathcal{K}_m}((W, \varphi), (W', \varphi'))$ and $[\psi', n'] \in Mor_{\mathcal{K}_m}((W', \varphi'), (W'', \varphi''))$ is defined by

$$[\psi', n'] \circ [\psi, n] = [\psi' \circ \psi, n' + n]$$

One can easily see that $\mathcal{K}_m$ is a well-defined category and $[id_W, 0]$ is the identity morphism over any object (W, φ) in $\mathcal{K}_m$. For a given object (W, φ)

in the $\mathcal{K}_m$ category by $[W, \varphi]$ we shall denote the class of all objects in $\mathcal{K}_m$ isomorphic to it.

The construction sketched above is natural in the following sense. Let $G : \mathcal{K} \to \mathcal{L}$ be a functor. Then there exists the induced functor $G_m : \mathcal{K}_m \to \mathcal{L}_m$ defined as follows.

$$G_m(W, \varphi) = (G(W), G(\varphi)),$$

$$G_m([\psi, n]) = [G(\psi), n]$$

for any object (W, φ) and morphism $[\psi, n]$ in $\mathcal{K}_m$. Clearly, G_m is covariant if G is covariant and contravariant if G is contravariant. Let us note two simple properties of the $\mathcal{K}_m$ category now.

Proposition 1.1 1°. *If $\iota : W \to W'$ and $\omega : W' \to W$ are morphisms in $\mathcal{K}$ then the objects $(W, \omega \circ \iota)$ and $(W', \iota \circ \omega)$ are isomorphic in $\mathcal{K}_m$.*
2°. *If $\varphi : W \to W$ and $\varphi' : W' \to W'$ are isomorphisms in $\mathcal{K}$ and (W, φ) and (W', φ') are isomorphic in $\mathcal{K}_m$ then φ and φ' are conjugate by an isomorphism.*

Proof.

1°. One can easily see that

$$[\iota, 0] : (W, \omega \circ \iota) \longrightarrow (W', \iota \circ \omega)$$

and

$$[\omega, 1] : (W', \iota \circ \omega) \longrightarrow (W, \omega \circ \iota)$$

are reciprocal isomorphisms.
2°. Let $[\psi, n] : (W, \varphi) \to (W', \varphi')$ and $[\psi', n'] : (W', \varphi') \to (W, \varphi)$ be reciprocal isomorphisms. Then,

$$[id_W, 0] = [\psi', n'] \circ [\psi, n] = [\psi' \circ \psi, n' + n].$$

Hence there exists $k \in \mathbf{Z}^+$ such that

$$\varphi^{n+n'+k} = \psi' \circ \psi \circ \varphi^k.$$

Since φ is an isomorphism, $\varphi^{n+n'} = \psi' \circ \psi$. Similarly, $\varphi'^{n+n'} = \psi \circ \psi'$. It follows that ψ has both right and left inverses and hence it is an isomorphism. By the definition of morphisms in $\mathcal{K}_m$, $\psi \circ \varphi = \varphi' \circ \psi$. □

Let us recall the definition of the Conley index based on the above construction now. For the details, the reader is referred to [13]. Other constructions of Conley indices can be found in [6], [8] and [10]. Let X be a locally compact metric space and $f : X \to X$ a continuous map. A pair $Q = (Q_1, Q_0)$ of compact subsets of X is called an index pair for an isolated invariant set S with respect to f if and only if $S = Inv_f cl(Q_1 \setminus Q_0) \subset int(Q_1 \setminus Q_0)$, Q_0 is positively invariant in Q_1 (i.e. $f(Q_0) \cap Q_1 \subset Q_0$) and Q_0 is an exit set for Q_1 (which means that $f(Q_1 \setminus Q_0) \subset Q_1$). For such Q there exists a continuous map $f_Q : Q_1/Q_0 \to Q_1/Q_0$ induced by f which will be called the index map. It maps $[Q_0]$ into itself and therefore can be treated as a map of the pointed space $(Q_1/Q_0, [Q_0])$ into itself. We shall call an index pair Q regular if and only if $(f_Q)^{-1}(\{[Q_0]\})$ is a neighborhood of $[Q_0]$. This definition is less restrictive than that in [4] or [5].

Let S be an isolated invariant set with respect to a continuous map $f : X \to X$. The Conley index of S, denoted by $h(S, f, X)$ is defined as the class of all objects isomorphic to $((Q_1/Q_0, [Q_0]), f_Q)$ in $\mathcal{H}top_m$. In [13] it is proved that this class is independent on the choice of Q. Thus,

$$h(S, f, X) = [(Q_1/Q_0, [Q_0]), f_Q].$$

The naturality of the construction of the category of objects equipped with a morphism allows to define the cohomological Conley index and the q-dimensional cohomological Conley index, denoted by $h^*(S, f, X)$ and $h^q(S, f, X)$ in the following way.

$$h^*(S,f,X) = (H^*)_m(h(S,f,X)),$$

$$h^q(S,f,X) = (H^q)_m(h(S,f,X)).$$

In the sequel, 0 will denote the zero isomorphism classes of objects in each of the $\mathcal{K}_m$ categories where $\mathcal{K} = \mathcal{V}, \mathcal{V}_G, \mathcal{H}top$ (i.e. we put $0 = [W, \varphi]$ where W is the pointed one-point space or the zero (graded) vector space, according to the case and φ is the only morphism of W into itself).

The Conley index defined above is slightly different from the one introduced in [6], [8] or [10]. However, the proofs of the basic properties of the indices of Conley type, like the continuation, commutativity and the Ważewski property carry over directly to our case (compare proposition 1.1 above and theorem 1.4 in [8]).

2. *Notation and main results*

Let X be a locally compact metric space and $f : X \to X$ a continuous map. N will denote an isolating neighborhood for an isolated invariant set S with respect to f, decomposing into a disjoint union of compact sets N_i, $i = 1, 2, \ldots, k$. For a set $Z \subset \{1, 2, \ldots, k\}$ we put

$$N_Z = \bigcup_{i \in Z} N_i.$$

By Σ we denote the set of all subsets of $\{1, 2, \ldots, k\}$. For all $l \in \mathbf{N}$ and $\tau = (Z_0, Z_1, \ldots, Z_{l-1}) \in \Sigma^l$ we define

$$N_\tau = \bigcap_{j=0}^{l-1} f^{-j}(N_{Z_j}).$$

Put $S = Inv_f N$. In the next section we shall prove the following

Proposition 2.1 *N_τ is an isolating neighborhood with respect to f^l. Furthermore,*

$$Inv_{f^l} N_\tau \subset S.$$

In the sequel, we shall write S_τ for $Inv_{f^l} N_\tau$. The following theorem relates the Conley indices of these sets and shows that, in fact, they are determined by a finite number of data. The proof will be given in the next section.

Theorem 2.1 *There exists a graded vector space $V^* = \{V^q\}_{q\in\mathbf{Z}}$ over F and graded linear endomorphisms $\varphi_i^* = \{\varphi_i^q\}_{q\in\mathbf{Z}}$ of V^* $(i = 0, 1, \ldots, k)$ such that for each $l \in \mathbf{Z}$ and $\tau = (Z_0, Z_1, \ldots, Z_{l-1}) \in \Sigma^l$,*

$$h^*(S_\tau, f^l, X) = [V^*, \varphi_\tau^*], \tag{2.1}$$

where $\varphi_\tau^* = \varphi_{Z_0}^* \circ \varphi_{Z_1}^* \circ \ldots \circ \varphi_{Z_{l-1}}^*$ and $\varphi_Z^* = \sum_{i\in Z} \varphi_i^*$ for each $Z \in \Sigma$.

Now, consider the simplest case, $k = 2$. Then, N is a disjoint union of two compact sets N_1 and N_2. Suppose that the Conley index of S is trivial and the Conley index of the invariant part of N_1 is equal to the index of a hyperbolic fixed point (note that the standard Smale's horseshoe satisfies these assumptions - see [6]). In [4] it is conjectured that under the above assumptions there exists a semiconjugacy defined on S onto the phase space of the shift map. We shall prove an extended version of this conjecture, giving also some lower bound for the number of periodic orbits of f in S. In order to state it, we need the following definition.

Definition 2.1 *Let (V, φ) be an object in $\mathcal{V}_m$. There exists an $M \in \mathbf{N}$ such that for each $m \geq M$*

$$dim\ im\ \varphi^M = dim\ im\ \varphi^m \in \mathbf{Z}^+ \cup \{\infty\}.$$

The asymptotic dimension of (V, φ), denoted by $adim(V, \varphi)$ is defined as $dim\ im\ \varphi^M$.

We have the following proposition, which will be proved in section 4.

Proposition 2.2 *Isomorphic objects in $\mathcal{V}_m$ have equal asymptotic dimensions.*

It follows that the asymptotic dimension of an isomorphism class of objects in $\mathcal{V}_m$ can be defined as the common value of the asymptotic dimensions of each of its representatives. Put

$$\Pi^+ = \prod_{i=0}^{\infty} \{1, 2\}.$$

By σ we denote the shift map on Π^+.

Theorem 2.2 *Let N be an isolating neighborhood with respect to a continuous map f of a locally compact metric space X into itself. Suppose that N is a disjoint union of two compact sets N_1 and N_2. Then N_1 is an isolating neighborhood with respect to f. If, for some $r \in \mathbf{Z}$,*

$$adim\ h^r(Inv_f N, f, X) = 0 \qquad and \qquad adim\ h^r(Inv_f N_1, f, X) = 1$$

then there exists an $s \in \mathbf{N}$ and a continuous surjection $h : Inv_f N \to \Pi^+$ such that the following diagram commutes

$$\begin{array}{ccc} Inv_f N & \xrightarrow{\ f^s\ } & Inv_f N \\ \downarrow h & & \downarrow h \\ \Pi^+ & \xrightarrow{\ \sigma\ } & \Pi^+ \end{array} \tag{2.2}$$

If, in addition, X is an ENR, $F = \mathbf{Q}$ and $adim\ h^q(Inv_f N_1, f, X) = 0$ for each $q \neq r$ then h and s can be chosen in such a way that each periodic sequence in Π^+ is an image of a periodic point of f^s with the same principal period.

As it can be seen, the above theorem provides information about the behavior of some iteration of f rather than about f itself. Roughly speaking, this is caused by the fact that its assumptions provide only a partial information about the φ_i^* endomorphisms which appear in theorem 2.1. Thus, a possible method of reducing s to 1 is to find these endomorphisms (we note that the proof of theorem 1.1 explains how to do this). Alternatively, one can use a refinement of the Conley index which takes into account the existence of a decomposition of the isolated invariant set into a number of compact subsets. A definition of such an index is given in [12].

3. *Proofs of the basic results*

Proof of proposition 2.1. Assume that $x \in S_\tau$, $\tau = (Z_0, Z_1, \ldots, Z_{l-1})$. There exists a sequence $\{x_n\}_{n\in\mathbf{Z}} \subset N_\tau$ such that $x_0 = x$ and $f^l(x_n) = x_{n+1}$ for each $n \in \mathbf{Z}$. Define a sequence $\{x'_n\}_{n\in\mathbf{Z}}$ as follows. For $d \in \mathbf{Z}$ and $t \in \{0, 1, \ldots, l-1\}$ put $x'_{dl+t} = f^t(x_d)$. By the definition of N_τ, $x'_{dl+t} \in N_{Z_t} \subset N$. Since N is an isolating neighborhood with respect to f and $f(x_n) = x_{n+1}$ for all $n \in \mathbf{Z}$,

$$\forall_{t\in\{0,1,\ldots,l-1\}}\ x'_t \in int(N) \cap N_{Z_t} = int(N_{Z_t}).$$

Thus,

$$x \in \bigcap_{t=0}^{l-1} f^{-t}(int(N_{Z_t})) \subset int(N_\tau).$$

The proof is finished. □

By corollary 2 in [5], there exists a regular index pair $Q = (Q_1, Q_0)$ for S with respect to f such that $Q_1 \subset N$. Let $Y = Q_1/Q_0$, $p = [Q_0] \in Y$, $g = f_Q : Y \to Y$ and $Y_i = \pi(Q_1 \cap N_i) \cup \{p\}$ for each $i \in \{1, 2, \ldots, k\}$ where $\pi : Q_1 \to Y$ is the projection map. For $Z \in \Sigma$ we put $Y_Z = \bigcup_{i\in Z} Y_i =$

$\pi(Q_1 \cap N_Z) \cup \{p\}$. For $x \in Q_1$, instead of writing $\pi(x)$ we shall often write briefly $[x]$. The continuous maps $r_Z, g_Z : Y \to Y$ are defined by

$$r_Z([y]) = \begin{cases} y & \text{if } \ y \in Y_Z \\ p & \text{otherwise} \end{cases} ,$$

$$g_Z = g \circ r_Z.$$

For $\tau = (Z_0, Z_1, \ldots, Z_{l-1}) \in \Sigma^l$ we put

$$g_\tau = g_{Z_{l-1}} \circ g_{Z_{l-2}} \circ \ldots \circ g_{Z_0}.$$

Notice that $g_Z(p) = g_\tau(p) = p$. Therefore, both g_Z and g_τ induce maps of the pointed space (Y, p) into itself, which will be denoted $\bar{g}_Z$ and $\bar{g}_\tau$, respectively. Put $A = g^{-1}(\{p\})$. Clearly, A is a compact neighborhood of p and $g_Z(A) = g_\tau(A) = \{p\}$ for all $Z \in \Sigma$ and $\tau \in \Sigma^l$.

Lemma 3.1 *For each $l \in \mathbf{N}$ and $\tau \in \Sigma^l$,*

$$h(S_\tau, f^l, X) = [(Y, p), \bar{g}_\tau].$$

Proof. We have the following formula for g_τ, $\tau = (Z_0, Z_1, \ldots, Z_{l-1}) \in \Sigma^l$.

$$g_\tau([x]) = \begin{cases} [f^l(x)] & \text{if } \ \forall_{i \in \{0,1,\ldots,l-1\}} \ f^i(x) \in N_{Z_i} \cap (Q_1 \setminus Q_0) \\ p & \text{otherwise} \end{cases} . \tag{3.1}$$

Let $U \subset Q_1 \setminus Q_0$ be a neighborhood of S_τ such that

$$\forall_{i \in \{0,1,\ldots,l-1\}} \ f^i(U) \subset N_{Z_i} \cap (Q_1 \setminus Q_0).$$

Take V a neighborhood of S_τ such that $V \cup f^l(V) \subset U$. By (3.1) the following diagram is commutative.

$$\begin{array}{ccc} V & \xrightarrow{\ f^l\ } & U \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \pi(V) & \xrightarrow{\ g_\tau\ } & \pi(U) \end{array}$$

Since $\pi(U)$ is open in Y and π restricted to U is a homeomorphism onto its image, $\pi(S_\tau)$ is an isolated invariant set with respect to g_τ and

$$h(\pi(S_\tau), g_\tau, Y) = h(S_\tau, f^l, X). \tag{3.2}$$

(this follows from the commutativity property of the Conley index, see [8], theorem 1.12). Clearly, the commutativity of the diagram implies that $\pi(S_\tau) \subset Inv_{g_\tau} cl(Y \setminus A)$. Let us show that the reverse inclusion holds. If $[x] \in Inv_{g_\tau} cl(Y \setminus A)$ then there exists a sequence $\{[x_n]\}_{n\in\mathbf{Z}} \subset Y \setminus \{p\}$ such that $x_0 = x$ and $g_\tau([x_n]) = [x_{n+1}]$ for all $n \in \mathbf{Z}$. By (3.1), $f^l(x_n) = x_{n+1}$ and $f^i(x_n) \in N_{Z_i} \cap (Q_1 \setminus Q_0)$ for all $n \in \mathbf{Z}$ and $i \in \{0, 1, \ldots, l-1\}$. It follows that $x_n \in \bigcap_{i=0}^{l-1} f^{-i}(N_{Z_i}) = N_\tau$ so that $x \in Inv_{f^l} N_\tau = S_\tau$. Hence $\pi(S_\tau) = Inv_{g_\tau} cl(Y \setminus A)$.

Since $\pi(S_\tau)$ does not intersect A, it is an isolated invariant set with respect to g_τ, having $cl(Y \setminus A)$ as its isolating neighborhood. One can easily see that (Y, A) is an index pair for $\pi(S_\tau)$ with respect to g_τ. Hence

$$h(\pi(S_\tau), g_\tau, Y) = [(Y/A, [A]), g'_\tau] \tag{3.3}$$

where $g'_\tau : (Y/A, [A]) \to (Y/A, [A])$ is the map induced by g_τ. Now, let $\kappa : (Y, p) \to (Y/A, [A])$ be the projection-induced map and $\omega : (Y/A, [A]) \to (Y, p)$ be induced by g_τ. By proposition 1.1,1°,

$$[(Y/A, [A]), g'_\tau] = [(Y/A, [A]), \kappa \circ \omega] = [(Y, p), \omega \circ \kappa] = [(Y, p), \bar{g}_\tau].$$

By (3.2) and (3.3) the proof is complete. □

For $i \in \{1, 2, \ldots, k\}$ instead of writing $r_{\{i\}}$ and $g_{\{i\}}$ we shall write briefly r_i and g_i. For any $i \in \{1, 2, \ldots, k\}$ and $Z \in \Sigma$, $l \in \mathbf{Z}^+$ and $\tau \in \Sigma^l$ the maps r_Z, r_i, g_Z, g_i and g_τ map the pair (Y, A) into itself. It will be convenient for us to denote the induced maps of this pair into itself by $\tilde{r}_Z$, $\tilde{r}_i$, $\tilde{g}_Z$, $\tilde{g}_i$ and

$\tilde{g}_\tau$, respectively. The proof of the following lemma can be obtained using the standard methods of cohomology theory (see for example [11], exercise 4.I.2).

Lemma 3.2 *For any $Z \in \Sigma$,*

$$H^*(\tilde{r}_Z) = \sum_{i \in Z} H^*(\tilde{r}_i).$$

Proof of theorem 2.1 By lemma 3.1,

$$h^*(S_\tau, f^l, X) = [H^*(Y,p), H^*(\bar{g}_\tau)].$$

Since $g_\tau(A) = \{p\}$, by applying proposition 1.1 in the same way as in the proof of lemma 3.1 we get

$$[H^*(Y,p), H^*(\bar{g}_\tau)] = [H^*(Y,A), H^*(\tilde{g}_\tau)].$$

By the definition of g_τ,

$$\tilde{g}_\tau^* = \tilde{g}_{Z_0}^* \circ \tilde{g}_{Z_1}^* \circ \ldots \circ \tilde{g}_{Z_{l-1}}^*.$$

By lemma 3.2, for each $Z \in \Sigma$,

$$\tilde{g}_Z^* = \sum_{i \in Z} \tilde{g}_i^*$$

so that (2.1) holds for $V^* = H^*(Y,A)$ and $\varphi_i^* = \tilde{g}_i^*$. $\square$

4. *Linear algebra*

This section deals with the purely algebraic notions which will be important in the proof of theorem 2.2. To begin with, let us prove proposition 2.2.

Proof of proposition 2.2. Let (V,φ) and (V',φ') be isomorphic objects of $\mathcal{V}_m$. Let $[\psi,n] : (V,\varphi) \to (V',\varphi')$ and $[\psi',n'] : (V',\varphi') \to (V,\varphi)$ be reciprocal isomorphisms. Put $m = n + n'$. Since $[\psi' \circ \psi, m] = [id_V, 0]$,

$$\psi' \circ \varphi'^s \circ \psi = \psi' \circ \psi \circ \varphi^s = \varphi^{s+m}$$

and therefore $dim\ im\ \varphi^{s+m} \leq dim\ im\ (\varphi')^s$ for sufficiently large $s \in \mathbf{Z}^+$. It follows that $adim(V, \varphi) \leq adim(V', \varphi')$. The reverse inequality can be proved in an analogous way. □

Definition 4.1 *An object (V^*, φ^*) in $(\mathcal{V}_G)_m = \mathcal{V}_{Gm}$ is called of finite type if there exists an object $(\bar{V}^*, \bar{\varphi}^*)$ in $\mathcal{V}_{Gm}$, isomorphic to (V^*, φ^*) and such that $\bar{V}^*$ is of finite type and $\bar{\varphi}^*$ is an isomorphism. In this case, the Lefschetz number of (V^*, φ^*) denoted by $\Lambda(V^*, \varphi^*)$ is defined as the ordinary Lefschetz number of $\bar{\varphi}^*$. By proposition 1.1,2°, it is independent of the choice of $(\bar{V}^*, \bar{\varphi}^*)$. An isomorphism class I of objects of $\mathcal{V}_{Gm}$ is called of finite type if and only if it has a representative being of finite type or, equivalently, each of its representatives is of finite type. Its Lefschetz number is defined as the Lefschetz number of any of its representatives and denoted by $\Lambda(I)$.*

Remark 4.1. Notice that the Leray reductions (see [6], [8]) of (V^*, φ^*) and $(\bar{V}^*, \bar{\varphi}^*)$ are isomorphic in the category $Auto(\mathcal{V}_G)$ (this follows from [13], theorem 6.1). Since the Leray reduction of the latter equals, up to an isomorphism, $(\bar{V}^*, \bar{\varphi}^*)$, it follows that the Lefschetz number in the sense of definition 4.1 is equal to the ordinary Lefschetz number of the Leray reduction.

The following three easy propositions deal with some technical details connected with the above definition.

Proposition 4.1 *If (V^*, φ^*) is an object in $\mathcal{V}_{Gm}$ and there exists an $M \in \mathbf{Z}^+$ such that the graded vector space $im\ (\varphi^*)^M$ is of finite type then (V^*, φ^*) is of finite type.*

Proof. Let $V_n^* = im\ (\varphi^*)^n$ and φ_n^* be the restriction of φ^* to V_n^*. By applying proposition 1.1,1° in the standard way (see the proof of lemma 3.1 and theorem 2.1) one proves inductively that (V^*, φ^*) and (V_n^*, φ_n^*) are isomorphic in $\mathcal{V}_{Gm}$. By assumptions, φ_n^* is an isomorphism for n sufficiently large. □

Proposition 4.2 *If (V^*, φ^*) is of finite type then there exists an $M \in \mathbf{Z}^+$ such that for all $q \in \mathbf{Z}$*

$$dim\ im\ (\varphi^q)^M = adim(V^q, \varphi^q).$$

Proof. We define the functor $S : \mathcal{V}_G \to \mathcal{V}$ in the following way. For a graded vector space U^* we put

$$S(U^*) = \bigoplus_{q \in \mathbf{Z}} U^q.$$

A graded linear map $\varphi^* : U^* \to W^*$ is mapped by S into the induced linear map of $S(U^*)$ into $S(W^*)$. By proposition 2.2 and definition 4.1, if (V^*, φ^*) is of finite type then $(V_0, \varphi_0) = S_m((V^*, \varphi^*))$ has finite asymptotic dimension. Thus, $dim\ im\ (\varphi_0)^M$ is finite and constant for M sufficiently large. By the definition of S, this means that

$$dim\ im\ (\varphi^q)^M = adim(V^q, \varphi^q)$$

for all $q \in \mathbf{Z}$. □

Lemma 4.1 *If (V^*, φ^*) is of finite type and there exists an $r \in \mathbf{Z}$ such that*

$$adim(V^q, \varphi^q) = \begin{cases} 0 & \text{if } q \neq r \\ 1 & \text{if } q = r \end{cases}$$

then $\Lambda(V^*, \varphi^*) \neq 0$.

Proof. Take an object $(\bar{V}^*, \bar{\varphi}^*)$ isomorphic to (V^*, φ^*) and such that $\bar{V}^*$ is of finite type and $\bar{\varphi}^*$ is an isomorphism. By proposition 2.2,

$$dim\ \bar{V}^q = adim(\bar{V}^q, \bar{\varphi}^q) = adim(V^q, \varphi^q).$$

Hence $\Lambda(V^*, \varphi^*) = (-1)^r tr(\bar{\varphi}^r) \neq 0$. □

The next lemma will be used in the proof of theorem 2.2.

Lemma 4.2 *Let V^* be a graded vector space and $\varphi_1^*, \varphi_2^* : V^* \to V^*$ be graded linear endomorphisms such that*

$$adim(V^r, \varphi_1^r) = 1 \qquad and \qquad adim(V^r, \varphi_1^r + \varphi_2^r) = 0$$

for some $r \in \mathbf{Z}$. There exists an $l \in \mathbf{N}$ and a sequence $(\sigma_0, \sigma_1, \ldots, \sigma_{l-1}) \in \{1,2\}^l$ such that $\sigma_a = 2$ for some $a \in \{0, 1, \ldots, l-1\}$ and for each $n \in \mathbf{Z}^+$ the graded endomorphisms

$$\psi_1^* = (\varphi_1^*)^{n+l} \quad and \quad \psi_2^* = (\varphi_1^*)^n \circ \varphi_{\sigma_0}^* \circ \varphi_{\sigma_1}^* \circ \ldots \circ \varphi_{\sigma_{l-1}}^* \tag{4.1}$$

satisfy the following condition.

For each $s \in \mathbf{N}$ and $(\xi_0, \xi_1, \ldots, \xi_{s-1}) \in \{1,2\}^s$,

$$[V^*, \psi_{\xi_0}^* \circ \psi_{\xi_1}^* \circ \ldots \circ \psi_{\xi_{s-1}}^*] \neq 0 \tag{4.2}$$

If, additionally, (V^*, φ_1^*) is of finite type and $adim(V^q, \varphi_1^q) = 0$ for all $q \neq r$ then $(V^*, \psi_{\xi_0}^* \circ \psi_{\xi_1}^* \circ \ldots \circ \psi_{\xi_{s-1}}^*)$ is of finite type and has a nonzero Lefschetz number provided n is sufficiently large.

Proof. There exist $M, M' \in \mathbf{N}$ and a one-dimensional subspace $W \subset V^r$ spanned by a nonzero vector $v \in V^r$ such that $im\ (\varphi_1^r)^m = W$ for all $m \geq M'$ and $(\varphi_1^r + \varphi_2^r)^M = 0$. In particular,

$$(\varphi_1^r)^{M+M'}(v) = dv \tag{4.3}$$

for some nonzero $d \in F$. Let $\bar{1}$ be the sequence of length M whose all entries are 1. Notice that

$$\begin{aligned} 0 &= (\varphi_1^r)^{M'} \circ (\varphi_1^r + \varphi_2^r)^M(v) = \\ &= dv + \textstyle\sum_{(\nu_0,\nu_1,\ldots,\nu_{M-1}) \in \{1,2\}^M \setminus \{\bar{1}\}} (\varphi_1^r)^{M'} \circ \varphi_{\nu_0}^r \circ \varphi_{\nu_1}^r \circ \ldots \circ \varphi_{\nu_{M-1}}^r(v) \end{aligned}$$

Since $dv \neq 0$, at least one of the remaining summands on the right-hand side is nonzero. Since each of them is a member of W, this means that for some nonzero $e \in F$ and a sequence $(\nu_0, \nu_1, \ldots, \nu_{M-1}) \in \{1,2\}^M$, not equal to $\bar{1}$,

$$(\varphi_1^r)^{M'} \circ \varphi_{\nu_0}^r \circ \varphi_{\nu_1}^r \circ \ldots \circ \varphi_{\nu_{M-1}}^r(v) = ev \tag{4.4}$$

Put $l = M + M'$ and define a sequence $(\sigma_0, \sigma_1, \ldots, \sigma_{l-1}) \in \{1,2\}^l$ as follows

$$\sigma_i = \begin{cases} 1 & \text{if } i \in \{0, 1, \ldots, M'-1\} \\ \nu_{i-M'} & \text{if } i \in \{M', M'+1, \ldots, l-1\} \end{cases} .$$

It is clear from (4.3) and (4.4) that any composition of ψ_1^r and ψ_2^r defined by (4.1) in an arbitrary order takes a nonzero value on v and therefore is nonzero for each $n \in \mathbf{N}$. Since its image is equal to W,

$$adim(V^r, \psi^r_{\xi_0} \circ \psi^r_{\xi_1} \circ \ldots \circ \psi^r_{\xi_{s-1}}) = 1$$

for all $(\xi_0, \xi_1, \ldots, \xi_{s-1}) \in \{1,2\}^s$. Since the asymptotic dimension of the zero object is zero, the proof of (4.2) is complete by proposition 2.2. To prove the second part, notice that proposition 4.2 implies that under the additional hypothesis, $\psi_1^q = \psi_2^q = 0$ for all $q \in \mathbf{Z}$, $q \neq r$, provided n is large enough. Hence, by proposition 4.1, $(V^*, \psi^*_{\xi_0} \circ \psi^*_{\xi_1} \circ \ldots \circ \psi^*_{\xi_{s-1}})$ is of finite type. Since its asymptotic dimension is zero in all dimensions except the r-th, lemma 4.1 implies that

$$\Lambda(V^*, \psi^*_{\xi_0} \circ \psi^*_{\xi_1} \circ \ldots \circ \psi^*_{\xi_{s-1}}) \neq 0. \quad \square$$

5. *Finiteness of type of the Conley index*

The aim of this section is to prove that the Conley index on ENR's is of finite type in the sense of definition 4.1. Our definition of finiteness of type is more restrictive than that in [9], since it requires not only the finiteness of type of the Leray reduction, but also some kind of finiteness of the reducing process. This property of the Conley index will be important in the proof of the part of theorem 2.2 concerning the existence of periodic points. The main tool in the proof is the following lemma.

Lemma 5.1 *Let S be an isolated invariant set with respect to a continuous map $f : \mathbf{R}^n \to \mathbf{R}^n$. There exists a polyhedral index pair for S.*

Proof. We shall make use of the basic concepts of the Conley index theory for multivalued maps, introduced in [2]. For $\epsilon > 0$ we put

$$\mathcal{A}(\epsilon) = \{A_1 \times A_2 \times \ldots \times A_n : \text{ for each } i \in \{1, 2, \ldots, n\} \text{ there exists } k \in \mathbf{Z} \text{ such that } A_i = \{k\epsilon\} \text{ or } A_i = [k\epsilon, (k+1)\epsilon]\}.$$

In what follows, a set $M \subset \mathbf{R}^n$ will be called ϵ-regular if it is a union of members of $\mathcal{A}(\epsilon)$. The multivalued maps $T_\epsilon, F_\epsilon : \mathbf{R}^n \to \mathbf{R}^n$ are defined by

$$T_\epsilon(x) = \bigcup \{A \in \mathcal{A}(\epsilon) : x \in A\},$$

$$F_\epsilon = T_\epsilon \circ f \circ T_\epsilon.$$

One easily checks that they are upper semicontinuous.

Let N be an isolating neighborhood for S. For ϵ small enough, $N_\epsilon = \bigcup\{A \in \mathcal{A}(\epsilon) : A \subset N\}$ is an isolating neighborhood with respect to F_ϵ in the sense of [2], containing S. Since f is a selector of F_ϵ, $S \subset S_\epsilon = Inv_{F_\epsilon} N_\epsilon$. By [2], theorem 2.6 there exists a pair (P_1, P_0) of compact subsets of N_ϵ satisfying the following conditions (cf [2],definition 2.5).

$$\forall_{i \in \{0,1\}} \; F_\epsilon(P_i) \cap N_\epsilon \subset P_i, \tag{5.1}$$

$$F_\epsilon(P_1 \setminus P_0) \subset N_\epsilon, \tag{5.2}$$

$$S_\epsilon \subset int(P_1 \setminus P_0). \tag{5.3}$$

Take $d \in \mathbf{N}$ and put

$$Q_i = \bigcup\{A \in \mathcal{A}(\epsilon/d) : A \subset P_i\}$$

for $i = 0, 1$. Clearly, Q_i are ϵ/d-regular and hence the pair (Q_1, Q_0) is polyhedral. We shall prove that for d large enough, (Q_1, Q_0) is an index pair for S with respect to f.

By (5.3), if d is large enough then $S \subset S_\epsilon \subset int(Q_1 \setminus Q_0)$. Hence

$$Inv_f cl(Q_1 \setminus Q_0) \subset Inv_f N = S \subset int(Q_1 \setminus Q_0).$$

Notice that $f(Q_0)\cap Q_1 \subset F_\epsilon(P_0)\cap N_\epsilon$. Since the set on the right-hand side is ϵ/d-regular and contained in P_0 by (5.1), it is contained in Q_0. It remains to prove that Q_0 is an exit set for Q_1. Take $x \in Q_1 \setminus Q_0$. Let A be the smallest (with respect to inclusion) set in $\mathcal{A}(\epsilon/d)$ for which $x \in A$. Then $A \subset P_1$. Since $x \notin Q_0$, there exists an $x' \in A$ such that $x' \notin P_0$. By the definition of F_ϵ, $F_\epsilon(x) \subset F_\epsilon(x')$. Hence, by (5.2),

$$f(x) \in F_\epsilon(x) \subset F_\epsilon(x') \subset F_\epsilon(P_1 \setminus P_0) \subset F_\epsilon(P_1) \cap N_\epsilon.$$

Since the set on the right-hand side is ϵ/d-regular and contained in P_1 by (5.1), it is contained in Q_1. The proof is finished. □

The next lemma concerns the finiteness of type of the Conley index and the relationship between the Conley index and fixed point index.

Lemma 5.2 *Let X be an ENR and $S \subset X$ an isolated invariant set with respect to a continuous map $f : X \to X$. Then, $h^*(S, f, X)$ is of finite type and, if $F = \mathbf{Q}$,*

$$ind(f, int(N)) = \Lambda(h^*(S, f, X))$$

where N is an isolating neighborhood for S with respect to f.

Proof. First, assume that X is an open subset of $\mathbf{R}^n$. Let $\bar{f} : \mathbf{R}^n \to \mathbf{R}^n$ be a map with a compact image such that f and $\bar{f}$ restricted to N are equal. Clearly, S is an isolated invariant set with respect to $\bar{f}$,

$$ind(f, int(N)) = ind(\bar{f}, int(N)) \quad \text{and} \quad h^*(S, f, X) = h^*(S, \bar{f}, \mathbf{R}^n). \quad (5.4)$$

By lemma 5.1, S admits a polyhedral index pair with respect to $\bar{f}$. By proposition 4.1, $h^*(S, \bar{f}, \mathbf{R}^n)$ is of finite type. Furthermore, by theorem 4 in [5] and remark 4.1,

$$\Lambda(h^*(S, \bar{f}, \mathbf{R}^n)) = ind(\bar{f}, int(N)).$$

To finish the proof, apply (5.4).

For the proof in the general case, take U an open subset of $\mathbf{R}^n$ and continuous maps $r : U \to X$ and $i : X \to U$ such that $r \circ i = id_X$. By the commutativity properties of the Conley and fixed point indices ([1],VII.5.16 and [8],theorem 1.11), $i(S)$ is an isolated invariant set with respect to $i \circ f \circ r$,

$$h^*(i(S), i \circ f \circ r, U) = h^*(S, f, X)$$

and

$$ind(i \circ f \circ r, r^{-1}(int(N))) = ind(f, int(N)).$$

Let $N' \subset r^{-1}(int(N))$ be an isolating neighborhood for $i(S)$. Then, since the lemma is proved for open subsets of $\mathbf{R}^n$, and all fixed points of $i \circ f \circ r$ in $r^{-1}(int(N))$ belong to $i(S)$,

$$\Lambda(h^*(i(S), i \circ f \circ r, U)) = ind(i \circ f \circ r, int(N')) = ind(i \circ f \circ r, r^{-1}(int(N))).$$

We sum up that $\Lambda(h^*(S, f, X)) = ind(f, int(N))$. $\square$

6. *Proof of theorem 2.2*

Below we use the notation introduced in section 2 with $k = 2$. It follows from theorem 2.1 that there exists a graded vector space V^* over F and graded linear endomorphisms $\varphi_1^*, \varphi_2^* : V^* \to V^*$ such that

$$h^*(S, f, X) = [V^*, \varphi_1^* + \varphi_2^*]$$

and for each $m \in \mathbf{N}$ and $\tau = (\{z_0\}, \{z_1\}, \dots, \{z_{m-1}\}) \in \Sigma^m$ ($z_j \in \{1, 2\}$ for $j = 0, 1, \dots, m-1$),

$$h^*(S_\tau, f^m, X) = [V^*, \varphi_{z_0}^* \circ \varphi_{z_1}^* \circ \dots \circ \varphi_{z_{m-1}}^*]. \tag{6.1}$$

By hypothesis, $adim(V^r, \varphi_1^r + \varphi_2^r) = 0$ and $adim(V^r, \varphi_1^r) = 1$. Thus, the assumptions of lemma 4.2 are satisfied. Let $l \in \mathbf{N}$, $(\sigma_0, \sigma_1, \dots, \sigma_{l-1}) \in \{1, 2\}^l$

and $a \in \{0, 1, \ldots, l-1\}$ satisfy the assertion of this lemma and $n \in \mathbf{N}$ be fixed. The functions $\Phi : N \to \{1, 2\}$ and $h : S \to \Pi^+$ are defined by

$$\Phi(x) = \begin{cases} 1 & \text{if } x \in N_1 \\ 2 & \text{if } x \in N_2 \end{cases},$$

$$h(x) = \left(\Phi(f^{n+a+(l+n)j}(x))\right)_{j=0}^{\infty}.$$

Clearly, each one is continuous and the diagram (2.2) commutes with $s = l + n$. We have to prove that h is onto. Since its image is compact, it sufficies to show that the image of h is dense in Π^+. Take a sequence $(\alpha_0, \alpha_1, \ldots, \alpha_{m-1}) \in \{1, 2\}^m$. We shall prove that there exists an $x \in S$ such that the first m coordinates of $h(x)$ are equal to $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$. Consider the sequence

$$\tau = (Z_0^0, \ldots, Z_{s-1}^0, Z_0^1, \ldots, Z_{s-1}^1, \ldots Z_0^{m-1}, \ldots, Z_{s-1}^{m-1}) \in \Sigma^{ms} \tag{6.2}$$

defined by

$$Z_j^i = \begin{cases} \{1\} & \text{if } \alpha_i = 1 \text{ or } j \in \{0, 1, \ldots, n-1\} \\ \{\sigma_{j-n}\} & \text{otherwise} \end{cases}. \tag{6.3}$$

By (6.1) and (4.2),

$$h^*(S_\tau, f^{ms}, X) = [V^*, \psi^*_{\alpha_0} \circ \psi^*_{\alpha_1} \circ \ldots \circ \psi^*_{\alpha_{m-1}}] \neq 0 \tag{6.4}$$

where ψ_i^* are defined by (4.1). By the Ważewski property of the Conley index (see [6], proposition 2.10, [8], proposition 1.9 or [13], remark finishing section 4), $S_\tau \neq \emptyset$. Take any $x \in S_\tau$. Then, by proposition 2.1, $x \in S$ and, by the definition of τ,

$$\forall_{i \in \{0,1,\ldots,m-1\}} \; f^{n+a+si}(x) \in N_{Z^i_{n+a}} = N_{\alpha_i}.$$

Hence the first m coordinates of $h(x)$ are $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$ and the proof of the first part of the theorem is finished.

Let us proceed to the proof of the part concerning periodic points now. Take a periodic sequence $\alpha = (\alpha_j)_{j=0}^{\infty} \in \Pi^+$ with principal period m. Under the additional hypothesis, (V^*, φ_1^*) is of finite type (by lemma 5.2) and

$adim(V^q, \varphi_1^q) = 0$ for all $q \neq r$. Lemma 4.2 and (6.4) imply that, for n large enough and τ defined by (6.2) and (6.3),

$$\Lambda(h^*(S_\tau, f^{ms}, X)) \neq 0.$$

By lemma 5.2, f^{ms} has a fixed point in S_τ. Clearly, this point is an m-periodic point of f^s. The commutativity of (2.2) implies that its principal period (with respect to f^s) is equal to m. $\square$

Acknowledgements

I wish to thank Professor Marian Mrozek for his encouragement to deal with the problems discussed in this paper.

References

[1] A.Dold, *Lectures on Algebraic Topology*, Springer-Verlag 1972.

[2] T.Kaczyński and M.Mrozek, 'Conley index for discrete multivalued dynamical systems', *to appear in Topology Appl.*.

[3] K.Mischaikow and M.Mrozek, 'Chaos in Lorenz Equations: a Computer Assisted Proof', *preprint.*

[4] K.Mischaikow and M.Mrozek, 'Isolating neighborhoods and chaos', *preprint.*

[5] M.Mrozek, 'Index pairs and the fixed point index for semidynamical systems with discrete time', *Fund.Math.133(1989),179-194.*

[6] M.Mrozek, 'Leray functor and cohomological Conley index for discrete dynamical systems', *Trans.Amer.Math.Soc.318(1990),149-178.*

[7] M.Mrozek, 'Open index pairs, the fixed point index and rationality of zeta functions', *Ergod.Th.Dyn.Sys.10(1990),555-564.*

[8] M.Mrozek, 'Shape index and other indices of Conley type for local maps on locally compact Hausdorff spaces', *to appear in Fund.Math.*.

[9] M.Mrozek, 'The Conley index on compact ANR's is of finite type', *Results Math.18(1990),306-313.*

[10] J.W.Robbin and D.Salamon, 'Dynamical systems, shape theory and the Conley index', *Ergod.Th.Dyn.Sys.8* (1988),375-393.*

[11] E.Spanier, *Algebraic Topology*, McGraw-Hill, New York 1966.

[12] A.Szymczak, 'The Conley index for decompositions of isolated invariant sets', *to appear in Fund.Math.*.

[13] A.Szymczak, 'The Conley index for discrete semidynamical systems', *to appear in Topology Appl.*.

Polish Academy of Sciences,

Institute of Mathematics,

ul.Abrahama 18,

81-825 Sopot,

Poland.

E-mail: matacs@halina.univ.gda.pl