

An Approximate Restatement of the Four Color Theorem

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Abstract

The celebrated Four Color Theorem was first conjectured in the 1850's. More than a century later Appel and Haken [1] were able to find the first proof. Previously, there had been many partial results, and many false proofs. Appel and Haken's famous proof has one "drawback": it makes extensive use of computer computation. More recently Robertson, Sanders, Seymour and Thomas [18] created another proof of the Four Color Theorem. However, their proof, while simplifying some technical parts of the Appel-Haken proof, still relies on computer computations.

There is some debate in the mathematical community about whether mathematicians should be satisfied with mathematical proofs that rely on extensive computation and there is interest in finding a new proof of the Four Color Theorem that relies on *no* computer computation. Such a proof would perhaps yield additional insights into why the Four Color Theorem is really true, and might yield new insights into the structure of planar graphs. In any case, there continues to be a search for such a proof.

The contribution of this paper is that we initiate a new approach towards proving the Four Color Theorem. Our approach is based on insights from computer science theory and modern combinatorics in the style of "Erdős". Tait proved in 1880 [20] that the Four Color Theorem is equivalent to showing that certain plane graphs have edge 3-colorings. Our main result is that this can be weakened to show that if every one of these plane graphs has even an *approximate edge 3-coloring*, then the Four Color Theorem is true. While it is not clear that our approach will necessarily open new attacks on the Four Color Theorem, the connections made in this paper seem to be of independent interest.

1 Introduction

The first proof to the celebrated Four Color Theorem (4CT) due to Appel and Haken [1] came a century after it was first conjectured in the 1850's. There had been many partial results and proof attempts since the late 19th century. Subsequently, Robertson et. al. [18] came up with a new proof based on the same approach, but simplifying some technical parts of [1]. The common “drawback” of both these proofs is that they rely on extensive computer computation.

A proof that did not use any computation would perhaps yield additional insights into why the 4CT is true, and might lead to better understanding of the structure of planar graphs. The contribution of this paper is that we initiate a new approach towards the search of such a proof. Our approach is based on insights from computer science theory and modern combinatorics in the style of “Erdős”. Tait proved in 1880 [20] that the 4CT is equivalent to showing that certain planar graphs have edge 3-colorings. Our main result is that if every one of his planar graphs has even an *approximate edge 3-coloring*, then the 4CT is true. More precisely, Tait [20] proved the following theorem:

Theorem 1.1. *Let G be a cubic two-edge connected plane graph. G has an edge 3-coloring if and only if G has a face 4-coloring;*

Tait’s proof of his theorem is constructive. In modern terms he actually proves that each implication can be done in linear time. Thus, if a graph G has an edge 3-coloring, then the face 4-coloring can be constructed in linear time. This fact will be used later on in our work.

Tait then used his theorem to claim a “proof” of the 4CT. He assumed that every two-edge connected cubic planar graph is Hamiltonian. This assumption is unfortunately false as shown by Tutte [21]: there are such planar graphs without any Hamiltonian tours. Note, the existence of such a tour easily proves that the graph is edge 3-colorable. This would show that it has a face 4-coloring. The proof of this is simple: Color the edges of the tour alternatively red/green. Color all other edges blue. It is easy to see that this is a valid edge 3-coloring of the graph.

In the hope of proving the 4CT, multiple restatements to the theorem have been established in [2, 4, 8, 9, 10, 13, 14, 19].

Appel and Haken [1] used an unavoidable set of configurations of every planar triangulation and gave the first computer-aided proof. This approach had been looked at since the beginning of the 20th century and had been used to prove several restricted versions of the theorem. Their version of the proof engaged a computer in 1200 hours of computation on the 1970’s vintage computers, which is mostly used up by verifications of various configurations.

Most recently, Robertson et. al. [18] came up with another version of a computer-aided proof, cutting the number of cases being tested to 633 from 1500. The paper also introduces some technical simplifications. Moreover, it gives a quadratic time algorithm to 4-color planar graphs, an improvement over the cubic time algorithm of Appel and Haken. Both these proofs used certain restatements of the theorem. Presently both of them are computer-aided, and hence impossible to humanly verify.

There continues to be considerable interest in finding a proof of the theorem that does not use any computer search. Even recently, new restatements of the theorem have been discovered.

Matiyasevich [15] proved that the 4CT is equivalent to the statement that some Diophantine equation $P(x_1, \dots, x_n) = 0$ has no solution in natural numbers x_1, \dots, x_n . He also showed in [16] another relation of the 4CT to associating probabilistic space on a triangulation of sphere, and

defining random events. The theorem is equivalent to different statements about positive correlation among certain pairs of these events.

In the same spirit, our paper introduces new restatements of the 4CT, similar to Tait’s, but weakening the requirements to imply the 4CT. To our knowledge, this is the first restatement to an “Erdős-type” *approximate* combinatorial problem. This provides hope of using new *computer science type* approximation techniques. One way of viewing our result is that to prove the 4CT, it is sufficient to prove an approximate version of Tait’s conjecture.

For a possible approach, we know that Brook’s theorem [5] tells us that every cubic graph is edge colorable using 3 or 4 colors. Tait needed that every cubic planar graph, that is also two-edge connected, be edge colorable using just 3 colors. One way of viewing our result is that to prove 4CT, it is sufficient to find an algorithm to edge 4-color two-edge connected cubic planar graphs such that the 4th color is used on $o(n)$ edges.

2 Preliminaries and Our Results

Definition An *edge 3-coloring* of a graph G is a mapping of edges of G to colors $\in \{1, 2, 3\}$. A *valid* edge 3-coloring is an edge 3-coloring such that no two edges incident on a common vertex map to the same color.

Definition A *face 4-coloring* of a plane graph G is a mapping of the faces of G to colors $\in \{1, 2, 3, 4\}$. A *valid* face 4-coloring is a face 4-coloring such that no two faces sharing a common edge map to the same color.

Definition v is an *error-vertex* in an edge 3-coloring of a cubic plane graph G if two edges incident on v map to the same color. The number of errors in a coloring of G is the number of error-vertices.

Our approach asks for *approximate* coloring, the approximation referring to errors rather than the number of colors, which might be of independent interest. Specifically, we ask for an algorithm for coloring 3-colorable graphs, with an error on at most ϵn vertices. What is the best ϵ achievable in polynomial time?

Since vertex-coloring 3-colorable graphs with 3 colors is NP-hard, many versions of approximate graph coloring have been considered in computer science literature. Two versions studied extensively are minimizing the number of colors (see [22, 3, 12], for example) and maximizing the number of non-monochromatic edges (unweighted MAX k -CUT; see [17, 6], for example). The problem we are considering is reminiscent of the unweighted version of MIN k -PARTITION problem whose objective is to minimize the number monochromatic edges. Kann et. al. [11] showed that for $k > 2$ and for every $\epsilon > 0$, there exists a constant α such that MIN k -PARTITION problem cannot be approximated within $\alpha|V|^{2-\epsilon}$, even for dense graphs. $O(\log n)$ -approximation algorithm for $k = 2$ and ϵn^2 -approximation algorithm, for any $\epsilon > 0$, for $k = 3$ are given in [7] and [11] respectively.

Our question is different from the MIN k -PARTITION problem in that we do not compare the solution to the optimal solution (that minimizes the number of error-vertices possible); rather we want to compare it to the number of vertices in one case and to the graph diameter in another. This question is not directly implied by any of the known results. In fact, it is not obvious that the results above imply anything even for the “edge-coloring” version of the MIN k -PARTITION problem. We believe that such new approximate colorings are interesting and important for other applications as well.

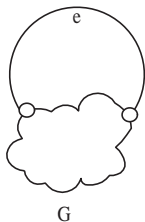


Figure 1: Graph Representation

The formal statement of the Four Color Theorem is as follows.

Theorem 2.1 (Four Color Theorem). *Every plane graph is face 4-colorable.*

Note that Tait [20] showed that to prove Theorem 2.1, it is sufficient to prove the same fact for two-edge connected cubic plane graphs. Throughout this paper, we use G to denote any cubic, two-edge connected, plane graph; even when we do not mention these conditions explicitly. We use Figure 1 to represent such G . Here e is any outer-edge in G .

Our Approach and Results: We initiate a more computer science type approach and provide *almost-tight* relationship between *approximate* versions of Tait’s conjecture and the 4CT.

We show an equivalence between the 4CT, and obtaining *approximate* edge 3-coloring of two-edge connected, cubic, planar graphs. Our results show that it would suffice to provide an edge coloring approach that makes an error on $o(|V(G)|)$ vertices on graphs G with vertices $V(G)$, to prove the four color theorem. Alternatively, it is sufficient to have an edge coloring algorithm for all G that makes an error on $2^{o(\text{diam}(G))}$ vertices, where $\text{diam}(G)$ stands for the diameter of G .

The two main contributions of our paper are that the following two theorems. We restate them later in the paper before proving them.

Theorem 3.1 (Size Bound Theorem). *Every two-edge connected, cubic, planar graph G can be edge 3-colored with $o(|V(G)|)$ errors \implies Four Color Theorem.*

Note that in the above theorem we regard the four color theorem as a conjecture (otherwise, the theorem statement is trivial). One way of looking at this theorem is we ask for a proof of the fact that every two-edge connected cubic planar graph G , on n vertices, can be decomposed into a union of disjoint cycles covering all vertices, with at most $o(n)$ odd length cycles. If such a decomposition exists, we can color the edges on every cycle with *red* and *green* alternatively, and color all other edges with *blue*. This way, there will be at most $o(n)$ errors. In fact, we do not even need a cycle cover, it is sufficient to have a cover of paths and cycles with $o(n)$ paths and odd cycles; see Corollary 3.4. This is a weaker requirement than Tait’s, as he needed a Hamiltonian cycle.

Theorem 4.1 (Diameter Bound Theorem). *For some constants d_0 and $\epsilon_0 < \frac{1}{2}$, every two-edge connected, cubic, planar graphs G with $\text{diam}(G) > d_0$ can be edge 3-colored with $\leq 2^{\epsilon_0 \text{diam}(G)}$ errors \implies Four Color Theorem.*

Cubic graphs have diameter at least $\log n$. So the statement of Theorem 4.1 is strictly weaker than Theorem 3.1 for graphs G with $\log n \leq \text{diam}(G) \leq \frac{1}{\epsilon_0} \log o(n)$. However, for graph with

$\text{diam}(G) > \frac{1}{\epsilon_0} \log o(n)$ (for example, graphs with diameter $\geq 2 \log n$, we do not need to prove anything if we are using Theorem 4.1 but would still need to prove $o(n)$ bound if we are using Theorem 3.1. Therefore, both these theorems provide different results, and the hope is that one of them will eventually help prove the 4CT.

It is important to note, however, that our technique does not provide fast algorithms for actually obtaining the face coloring. Rather, our proof is existential. The previous proofs gave polynomial time algorithms to compute the face 4-coloring, such as $O(n^2)$ time by [18]. The fact that we prove less is encouraging in an interesting way, as it may be possible to obtain simpler proofs.

While we talk about approximate edge colorings, an interesting problem is a discussion on approximate face 4-colorings as well. Tait's construction gives a local approach to obtaining an edge 3-coloring from a face 4-coloring, where the color of an edge is determined by the two faces it is adjacent to. Using this, one can convert a face 4-coloring with f edge errors into an edge 3-coloring with at most $2f$ vertex errors. Such an edge coloring can be obtained by coloring every non-error edge using Tait's algorithm and coloring error edges using one of the three colors arbitrarily. The converse is not implied by Tait's construction as his conversion from an edge 3-coloring to a face 4-coloring requires looking at the cycles induced by every pair of two colors. If the edge 3-coloring has errors, the induced subgraph of any pair of two colors need not form cycles. In particular, this implies that if we can find an algorithm to face 4-color graphs G with $o(n)$ errors, using our result, we can obtain a valid face 4-coloring of G .

Simple proofs of the fact that every plane graph can be face colored using 5 colors (Five Color Theorem) are known [5]. Our result asks whether there is an algorithm to obtain (or a simple proof of existence of) such a face 5-coloring with one of the colors being used $o(n)$ times. If there is such an algorithm, we could replace this fifth color with any of the other four colors, resulting in $o(n)$ errors in the face coloring, and thereby proving the Four Color Theorem, using our result.

3 Error allowed in the size of the graph

Theorem 3.1. *Every two-edge connected, cubic, planar graph G can be edge 3-colored with $o(|V(G)|)$ errors \implies Four Color Theorem.*

Proof. The condition that every two-edge connected cubic planar graph G can be edge 3-colored with $o(|V(G)|)$ errors is equivalent to the following statement.

For every ϵ , there exists n_0 such that every cubic two-edge connected planar graph with $n \geq n_0$ vertices is edge 3-colorable with at most ϵn errors. (1)

Assume (1) holds. Let G be any two-edge connected cubic plane graph with n vertices and let e be any edge that bounds the outer face of G . We show that G is face 4-colorable.

The main idea in the proof is to construct an amplified graph with many copies of G and select a very small ϵ so that we can isolate at least one copy of G that has no errors. This however requires that for our choice of ϵ , our amplified graph contains more vertices than the corresponding n_0 . For a concrete example, if we pick any $\epsilon < 1/n$ and if it turns out that the corresponding n_0 is at most $|V(G)|$ then there is no error in G by (1). However, one needs to be careful when $n_0 > |V(G)|$. For this case, we need to connect many copies of G to form the amplified graph G' while maintaining two-edge connectivity and the cubic and planar properties. So, it is not sufficient to have many disconnected copies of G in G' . Once we know that the constructed G' has more copies of G than

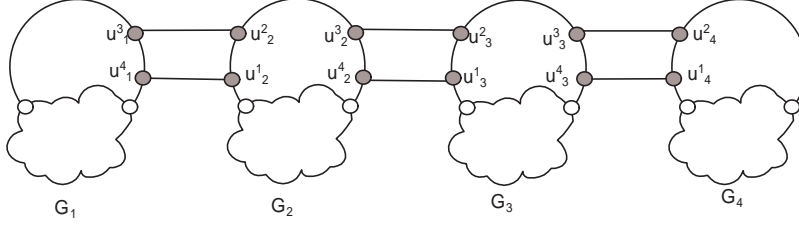


Figure 2: An example of constructing G' from four copies of G .

the number of errors, we again need to carefully *extract* one copy of G which has no errors in its edge coloring. However, this step again needs care, as the extracted G may have dangling edges and extra vertices that cannot be ignored. As shown in the proof, a *mirror trick* helps in finally obtaining a face 4-coloring of G . We first describe the main construction.

Line construction: Select $\epsilon = \frac{1}{n+4}$. By (1), there is a corresponding integer n_0 such that any cubic planar graph with at least n_0 vertices is edge 3-colorable with at most ϵn errors.

We construct a planar graph G' from n_0 copies of G as follows (see Figure 2). For $i = 1, 2, \dots, n_0$, let G_i be a copy of G with an outer edge $e \in E(G)$ subdivided two times for $i = 1$ and n_0 , and four times for other values of i . Denote the new vertices in G_1 by u_1^3 and u_1^4 , in G_{n_0} by $u_{n_0}^1$ and $u_{n_0}^2$, and in G_i , for other values of i , by $u_i^1, u_i^2, u_i^3, u_i^4$. For any $1 \leq i < n_0$, add an edge $u_i^3 u_{i+1}^2$ and $u_i^4 u_{i+1}^1$.

Claim 3.2. *There is an edge 3-coloring of G' such that one of the subdivisions of G_i along with its incident edges u_i^1, \dots, u_i^4 has no errors.*

Proof. Since we add one edge to each new vertex, G' is cubic. It is also clear from Figure 2 that G' is planar. Next we claim that G' is two-edge connected. Observe that G_i is edge-two connected for all i since G_i is a subdivision of G . We note that G' is connected since G_i is connected for all i and there are two edges connecting G_i and G_{i+1} for all i . Now suppose that there exists an edge e such that $G' \setminus e$ is disconnected. There are two cases. First, if e is an edge of G_i for some i then $G_i \setminus e$ is connected, since G_i is two connected, and so $G' \setminus e$ is also connected. Second, if e is an edge connecting G_i and G_{i+1} for some i then G_i and G_{i+1} are still connected since there are two edges connected between G_i and G_{i+1} . Therefore, $G' \setminus e$ is also connected.

We now apply (1) on G' . Since $|V(G')| = n_0(n+4) - 4 \geq n_0$, there exists an edge 3-coloring C of G' such that there are at most $\epsilon |V(G')| = \frac{n_0(n+4)-4}{n+4} < n_0$ errors. Therefore, there exists $1 \leq i \leq n_0$ such that C colors G_i with no error vertices. G_i with its dangling edges is the desired graph. \square

Let H^* be the G_i that has no errors. Let H be the union of H^* and the four dangling edges incident on u_i^1, \dots, u_i^4 . For an example, see Figure 3.

Although we know that H has no edge errors, it is not clear how one can obtain a valid face 4-coloring for G from this edge coloring. One cannot directly apply Tait's theorem as this graph is neither two-edge connected, nor cubic. Suppose there is an edge 3-coloring C of H such that all dangling edges have the same color, we can connect these edges in pairs to obtain a two-edge connected cubic planar H' that is edge 3-colored and contains G as a minor. However, one cannot

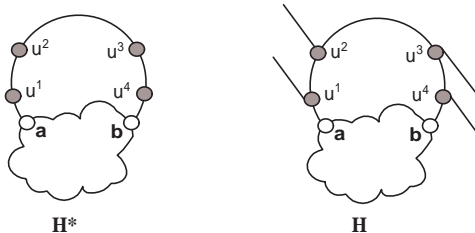


Figure 3: An example of H^* and H .

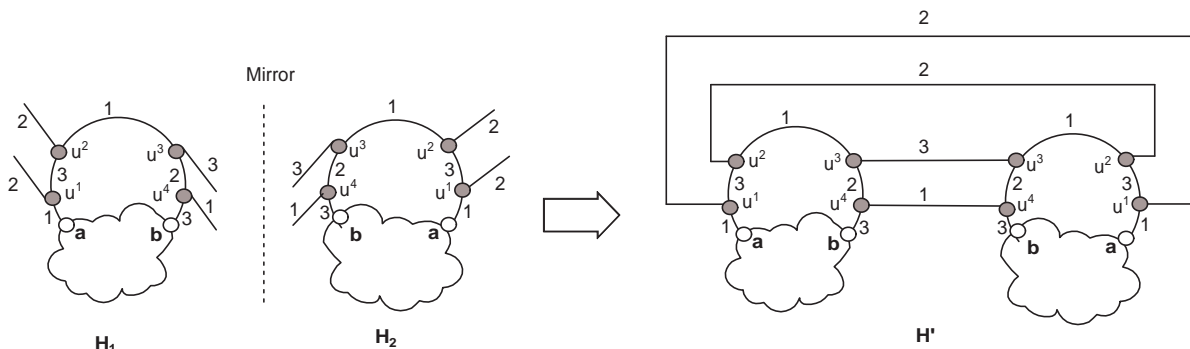


Figure 4: An example of extending an edge 3-coloring.

necessarily match up the four dangling edges incident on u_i^1, \dots, u_i^4 , in pairs, and still retain a valid edge 3-coloring, as we do not know what colors these four edge have received. The key idea in our proof is to perform a *mirror trick* by which we duplicate H and match up the dangling edges in the two copies of H . This way, we obtain a valid edge 3-coloring of this cubic two-edge connected planar graph. At this stage, we can apply Tait's theorem to obtain a valid face 4-coloring. Then one needs to argue that one can obtain a face 4-coloring for G from this coloring; it is in this step that we crucially use our method of connecting G 's, where we subdivided only one outer edge of each copy of G .

Claim 3.3. H^* is face 4-colorable.

Proof. Suppose that H has k dangling edges e_1, e_2, \dots, e_k (where k is either 2 or 4). Let C be an edge 3-coloring of H . Let H_1 and H_2 be two copies of H but let H_2 be drawn in such a way that it is a mirror (or horizontal flip) of H . We construct a cubic plane graph H' by connecting together each pair of edges e_i in H_1 and H_2 . Extend C to an edge 3-coloring of H' by painting each edge in H' the same color painted to the corresponding edge in H . (See Figure 4.) Note that H' is cubic and planar. We also claim that H' is two-edge connected since H_1 and H_2 are two-edge connected and $k \geq 2$. Therefore, H' has a face 4-coloring C' by Tait's theorem. We obtain H^* by deleting H_2 from H' . Obtain a face 4-coloring of H^* by coloring every internal face the same way it is colored by C' and color the unbounded face the same color of unbounded face of H^* . To see that this is a valid face 4-coloring of H^* observe that any pair of adjacent faces of H^* are also adjacent

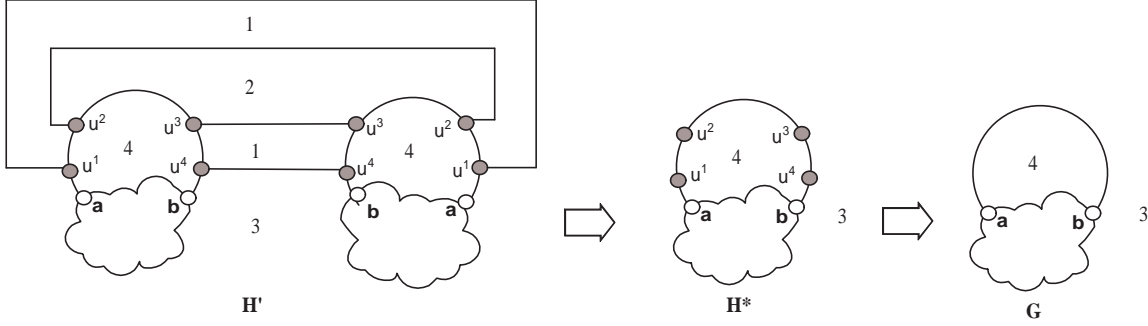


Figure 5: An example of a reaching a valid face 4-coloring of G from a valid edge 3-coloring of H' .

in H' . The crucial point is that the face originally in G adjacent to edge e is still adjacent to the *outer* face, as we had only subdivided the edge e , without drawing addition edges from the original vertices of e . \square

Since H^* is a subdivision of G , we can obtain a face 4-coloring of G from the face 4-coloring of H^* . For an example of how to obtain the face 4-coloring of G from the edge 3-coloring of H' , see Figure 5. \square

Although Theorem 3.1 is constructive, it does not provide a polynomial time algorithm. As seen in the proof, the algorithm for face 4-coloring G requires an approximate edge 3-coloring of G' .

Denote by $T_\epsilon(n)$ the time required to obtain an edge 3-coloring of an n node graph, with at most ϵn errors, for some fixed ϵ .

Time Complexity:

The time required to obtain a face 4-coloring of G is $T_\epsilon(n \cdot n_0)$, where $|G| = n$, $\epsilon = \frac{1}{n+4}$ and n_0 is the minimum size of the graph required corresponding to ϵ , as in the proof of Theorem 3.1. The number of copies of G in G' is also n_0 by our construction.

Notice that n_0 may be super-polynomial in n . The time required to isolate a G with no errors, once G' is colored, and the time to obtain the face coloring from it are $O(nn_0)$ and $O(n)$ respectively.

We now present a corollary to Theorem 3.1; this may provide new insights towards proving the 4CT.

Corollary 3.4. *Every two-edge connected cubic planar graph G whose vertices can be covered by vertex-disjoint paths and cycles such that there are $o(n)$ paths and odd cycles \implies four color theorem for G .*

Proof. Assume G is decomposed into a cover of vertex disjoint cycles and paths with $o(n)$ odd cycles and paths. For each of the cycles and paths in the cover, color the edges with 1 and 2 alternatively, starting with 1. Color all the remaining edges with 3. Since the graph is cubic, no two edges colored 3 share a vertex in common, except for the end points of the paths. Therefore, the only other errors occur when two edges colored 1 share a common vertex, in an odd cycle. Since

the number of errors in this coloring is exactly equal to the number of odd cycles plus twice the number of paths, which is $o(n)$, using Theorem 3.1, the corollary follows. \square

There are obvious ways of edge 3-coloring planar graphs (such as random coloring techniques) that make an error on a small constant fraction of all vertices. Our theorem raises the question whether there are better coloring algorithms that make an error on $o(n)$ vertices.

Since the above approach may require exponential time in retrieving the face 4-coloring, we now present an alternate approach which could yield a polynomial time algorithm. This, however, requires a stronger edge 3-coloring algorithm; one that makes an error of at most n^δ , for some $\delta < 1$ for all graphs G of size n .

Theorem 3.5. *Suppose for some fixed $0 < \delta < 1$, there exists an algorithm \mathcal{A} that edge 3-colors any cubic two-edge connected plane graph of size n with at most n^δ errors and runs in time $T(n)$ polynomial in n . Then, there exists an algorithm that face 4-colors any cubic two-edge connected plane graphs of size n with no errors and runs in time $T(n^{1/\delta})$.*

Proof. Let G be any cubic two-edge connected plane graph with n vertices and let $c = (2n)^{1/\delta-1} + 1$. Construct a graph G' with c copies of G using the method described in the proof of Theorem 3.1. Run \mathcal{A} on G' . Note that G' contains $c(n + 4) \leq 2cn = O(n^{1/\delta})$ vertices (we assume here that $n \geq 4$). So, it takes $O(n^{1/\delta})$ to construct G' and $T(n^{1/\delta})$ to run \mathcal{A} . The number of errors is at most $(2cn)^\delta < c$. Therefore, there exists a copy of G that contains no error vertices. Now, using Tait's algorithm on G'' defined as in the proof of Theorem 3.1, we obtain a face 4-coloring of G in time $O(n^{1/\delta})$. \square

4 Error allowed in the diameter of the graph

We now extend this technique to show that proving the four color theorem reduces to providing an edge coloring for all graphs with a bound on the errors as an exponential function of the diameter.

Theorem 4.1. *For some constants d_0 and $\epsilon_0 < \frac{1}{2}$, every two-edge connected, cubic, planar graph G with $\text{diam}(G) > d_0$ can be edge 3-colored with $\leq 2^{\epsilon_0 \text{diam}(G)}$ errors \implies Four Color Theorem.*

Proof. Suppose for all G with $\text{diam}(G) > d_0$, we have errors $\leq 2^{\epsilon_0 \text{diam}(G)}$ for some $\epsilon_0 < \frac{1}{2}$.

We use the tree construction in Figure 6 for this theorem. Let the original graph G have n vertices. We use the previous technique to expand this to a tree with h copies of G ; call this graph G' . For this construction, we have a binary tree with h leaf nodes. For every copy of G , we shall add three vertices by expanding an edge, and draw three edges from them. The central edge will connect to a leaf of the tree, while the left and right vertices will connect to the left and right leaf copies of G in the tree.

We choose h such that $h = \max\{2^{d_0}, 2^{\frac{2\epsilon_0 n}{1-2\epsilon_0}}\}$.

Notice that by our choice of h , it follows that $\text{diam}(G') > \log h \geq d_0$.

By construction of G' , we have,

$$\begin{aligned} \text{diam}(G') &\leq 2\text{diam}(G) + 2(\log h) \\ &\leq 2 \cdot n + 2(\log h) \end{aligned}$$

as the diameter of G is at most n .

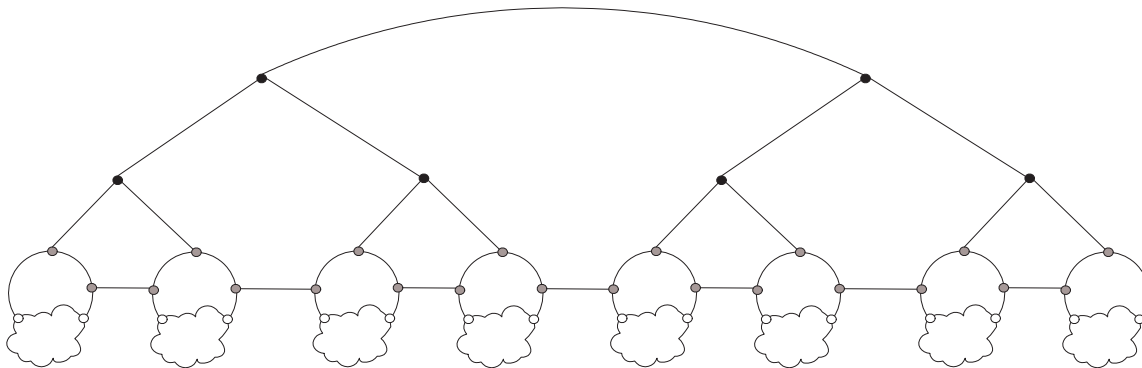


Figure 6: Tree construction

To use the previous technique of isolating one copy of G with no edge-coloring errors, we need that the number of copies of G be greater than the maximum possible number of errors. This condition is,

$$h > 2^{\epsilon_0 \cdot \text{diam}(G')}$$

which is implied by the following.

$$h > 2^{\epsilon_0 \cdot (2 \cdot n + 2 \log h)}.$$

Since we have $\text{diam}(G') > d_0$, we have errors $< 2^{\epsilon_0 \cdot (2 \cdot n + 2 \log h)}$.

So what we need is the following.

$$h > 2^{\epsilon_0 \cdot (2 \cdot n + 2 \log h)}$$

$$h > \{2^n\}^{\frac{2\epsilon_0}{1-2\epsilon_0}}$$

However, this condition is again true by our choice of h .

Therefore, we can isolate at least one copy of G with no error vertices. Using the technique in the proof of Theorem 3.1 (starting with the *mirror trick*), we can obtain a face 4-coloring of G . \square

Remark: For any cubic graph G , $\text{diam}(G)$ is at least $\log_2 n$. If for a graph G , the diameter is $\geq \frac{1}{\epsilon} \cdot \log n$, then the expression $2^{\epsilon_0 \text{diam}(G)}$ is at least n . So any edge coloring algorithm would satisfy this question. It follows that for a fixed ϵ_0 , we only need to consider graphs G with $\log n \leq \text{diam}(G) < \frac{1}{\epsilon} \cdot \log n$. Here $\log n$ is to base 2. Notice that our condition clearly holds if we allowed ϵ_0 to be 1. We require the condition to hold for some $\epsilon_0 < \frac{1}{2}$. It is also important to note that we cannot have a k -ary tree instead of a binary tree; this is because a k -ary tree would violate the cubic property of G' .

Time Complexity:

Let us denote by $T(x)$ the time taken to approximately Tait color graphs as required by this theorem, i.e., edge 3-color graphs G with $\text{diam}(G) > d_0$ with fewer errors than $2^{\epsilon_0 \text{diam}(G)}$. As

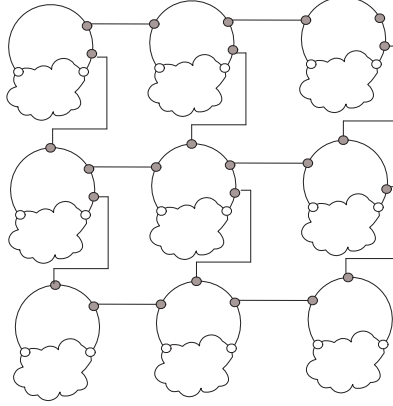


Figure 7: Grid construction

before, if we examine the proof of Theorem 4.1, we can get a constructive algorithm for face 4-coloring every cubic two-edge connected plane graph G . This algorithm runs in time $T(n2^{O(n)})$.

The reason this does not run in polynomial time is because the choice of h by the algorithm requires $h \geq 2^{\frac{2n\epsilon_0}{1-2\epsilon_0}}$, and the constructed graph G' is of size $O(nh)$.

Since this case, the edge coloring algorithm could take exponential time, we consider a different construction of G' that might be interesting on its own. We show by a *grid* construction that given a stronger edge 3-coloring algorithm, it is possible to obtain a fast face 4-coloring approach even via our technique.

Theorem 4.2. *Given an algorithm to edge 3-color every graph G with $< \text{diam}(G)^\delta$ errors in time $T(n)$ (where $|G| = n$, $\delta < 2$), there is an algorithm to face 4-color every graph G in time $T(n^{\frac{2+\delta}{2-\delta}})$.*

Proof. We use the grid construction for this theorem. Let the original graph, G that is assumed to be not four colorable, have n vertices and diameter $\text{diam}(G)$. We use the previous technique to expand this to an $h \times h$ grid, with h^2 copies of G ; call this graph G' (see Figure 7). We have, $|V(G')| = h^2 \cdot (n + 4)$ and $\text{diam}(G') \leq h \cdot n$

Suppose there exists an algorithm to edge 3-color G' with fewer errors than $(\text{diam}(G'))^\delta$. To use the above technique of isolating one copy of G with no edge-coloring errors, we need $(\text{diam}(G'))^\delta < h^2$ which is implied by $n^{\frac{\delta}{2-\delta}} < h$.

As long as $\delta < 2$, we can make h large enough to insure that the reduction goes through.

Time Complexity:

The constructed graph G' is of size $h^2 n = n^{\frac{2+\delta}{2-\delta}}$. So the time taken to edge 3-color G' with fewer than $\text{diam}(G)^\delta$ errors is $T(n^{\frac{2+\delta}{2-\delta}})$. The remaining steps of isolating a G with no errors and converting the edge coloring to a face coloring take time $O(n^{\frac{2+\delta}{2-\delta}})$ and $O(n)$ respectively, as before. \square

It is important to note that we do not need to make the grid too big. For instance, if $\delta = \frac{1}{2}$, i.e. one can edge 3-color with fewer than $\sqrt{\text{diam}}$ errors, then we only require that $h = n^{\frac{1}{3}}$. In this

case, the time required is $T(n^{\frac{5}{3}})$

5 Future directions and conclusions

We initiate a new approach towards proving the Four Color Theorem. Our results can be viewed as a generalization of Tait's [20]. We ask for edge colorings of edge 3-colorable graphs, with errors on any $o(n)$ vertices or $2^{\epsilon_0 \text{diam}(G)}$ vertices, where $\text{diam}(G)$ is the diameter of the graph, and ϵ_0 is any constant $< \frac{1}{2}$. The most interesting open question is whether there are simple proofs for the existence of, or algorithms for computing such edge 3-colorings as required by our two main Theorems 3.1 and 4.1. It might be interesting to improve the error bound in Theorem 4.1 to $o(2^{\text{diam}(G)})$. We would then need to only consider graphs with diameter $\log n + O(1)$ and prove an error bound of $o(n)$, thereby generalizing both theorems.

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