4- Point Interpolation

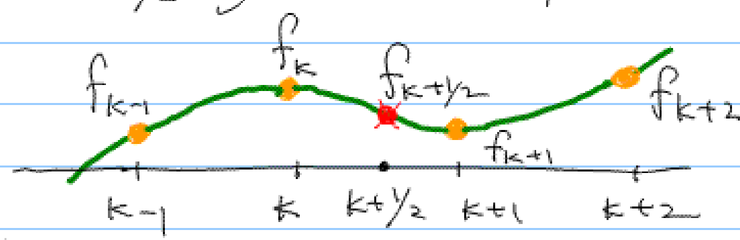
Note Title 3/24/2007

Let $f_0, f_1, f_2, \cdots f_n$ be samples of a function $f(t)$ at $t = 0, 1, 2, \cdots, n$, respectively.

We would like to double the number of samples, i.e., we would like to compute

$$f_{0+1/2},\ f_{1+1/2},\ f_{2+1/2},\ f_{3+1/2},\ \cdots,\ f_{n-1+1/2},$$

Consider computing $f_{k+1/2}$. We want to compute $f_{k+1/2}$ by an interpolation:



We first assume that the green curve is cubic.

$$f(t) = at^3 + bt^2 + ct + d$$

Then we solve the four equations:

$$f(k-1) = a(k-1)^3 + b(k-1)^2 + c(k-1) + d = f_{k-1}$$
$$f(k) = ak^3 + bk^2 + ck + d = f_k$$
$$f(k+1) = a(k+1)^3 + b(k+1)^2 + c(k+1) + d = f_{k+1}$$
$$f(k+2) = a(k+2)^3 + b(k+2)^2 + c(k+2) + d = f_{k+2}.$$

Since there are 4 equations and 4 unknowns a, b, c, d, we can solve for them.

With these a, b, c, and d, we can compute

$$f_{k+1/2} = f(k+1/2)$$
$$= -\frac{1}{16} f_{k-1} + \frac{9}{16} f_k + \frac{9}{16} f_{k+1} - \frac{1}{16} f_{k+2}.$$

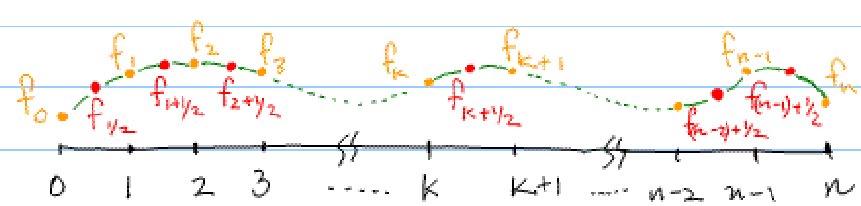
For example

$$f_{1+1/2} = -\frac{1}{16} f_0 + \frac{9}{16} f_1 + \frac{9}{16} f_2 - \frac{1}{16} f_3$$
$$f_{2+1/2} = -\frac{1}{16} f_1 + \frac{9}{16} f_2 + \frac{9}{16} f_3 - \frac{1}{16} f_4$$
$$\vdots$$

$f_{0+1/2}$ and $f_{(n-1)-1/2}$ are computed by repeating end samples:

$$f_{0+1/2} = -\frac{1}{16} f_0 + \frac{9}{16} f_0 + \frac{9}{16} f_1 - \frac{1}{16} f_2$$
$$f_{(n-1)-1/2} = -\frac{1}{16} f_{n-2} + \frac{9}{16} f_{n-1} + \frac{9}{16} f_n - \frac{1}{16} f_n$$

Thus, you can double the samples.



Once it is refined, you can apply this process multiple times, which will give you dense samples of $f(t)$.