[MATH2605] Exam 2

Name:____________________________________

Please print your full name.

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Problem 1 (20pt)

Let $\mathbf{R} = \frac{1}{9}\begin{bmatrix} 1 & 8 & 4 \\ 4 & -4 & 7 \\ 8 & 1 & -4 \end{bmatrix}$.

1.1 (15pt)

Compute the axis of rotation $\mathbf{u}$ and the angle of rotation θ.

1.2 (5pt)

Compute $\mathbf{Q}$ so that $\mathbf{Q}^n = \mathbf{R}$ (You do not have to simplify).

Problem 2 (20pt)

Consider an ODE $\ddot{x}+3\dot{x}+2x=kx$.

- When $k=0$, does $\lim_{t\to\infty} x(t)=0$? (5pt)
- When $k=3$, does $\lim_{t\to\infty} x(t)=0$? (5pt)
- Compute the range of k so that $\lim_{t\to\infty} x(t)=0$. (10pt)

Problem 3 (20pt)

Let $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. Consider the following ODE:

$$\dot{\mathbf{x}} = \begin{bmatrix} -x_1 + 2x_2^3 \\ -x_1 - x_2 \end{bmatrix}. \tag{1}$$

Let $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Using the forward Euler method with the time step $\Delta t = 0.1$, compute $\mathbf{x}(0.1)$, $\mathbf{x}(0.2)$, and $\mathbf{x}(0.3)$.

Problem 4 (20pt)

Let $\mathbf{x}(t) = \begin{bmatrix} 4\cos(t) \\ 4\sin(t) \\ 3t \end{bmatrix}$.

- Compute the velocity $\mathbf{v} = \dot{\mathbf{x}}$, the speed $v = |\mathbf{v}|$, and the acceleration $\mathbf{a} = \dot{\mathbf{v}}$.
- Compute the curvature κ by using $\kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{v^3}$.
- Compute the arc-length function $s(t)$, the tangent vector as functions of t and s, i.e., compute $\mathbf{T}(t)$ and $\mathbf{T}(s)$.
- Compute the curvature normal vector $\kappa\mathbf{N}$ by using $\kappa\mathbf{N} = \frac{d\mathbf{T}(s)}{ds}$.

5 points each.

Problem 5 (20pt)

5.1 (20pt)

Let $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$. Show that $e^{\mathbf{A}} = \mathbf{V}e^{\mathbf{D}}\mathbf{V}^{-1}$.

5.2 (This problem is for extra 5pt)

Let $\mathbf{A} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} = -2\mathbf{I} + \mathbf{N}$, where $\mathbf{I}$ is the identity matrix and $\mathbf{N} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Show that

$$e^{\mathbf{A}t} = e^{-2t\mathbf{I}+\mathbf{N}t} = e^{-2t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \tag{2}$$