[MATH2605] Homework 7

Due: Mar. 3

Problem 1, Curvature

Consider a parameterized curve.

$$\mathbf{x}(t) = \begin{bmatrix} \frac{4}{3}t^{3/2} \\ 2t\sin(t/2) \\ 2t\cos(t/2) \end{bmatrix}. \tag{1}$$

Compute $\mathbf{v}(t) = \dot{\mathbf{x}}, \quad \mathbf{a}(t) = \ddot{\mathbf{x}}, \quad v(t) = |\mathbf{v}(t)|, \quad \mathbf{T}(t) = \frac{\mathbf{v}}{v}$, and the arc length $s(t) = \int_0^t v(u)du$.

The curvature κ may be computed from the definition $\kappa = \frac{d\mathbf{T}(s)}{ds}$. However, to do this, the arc length parametrization $\mathbf{x}(s)$, and $\mathbf{T}(s)$ are required. In Problem 1, we study how to compute the curvature κ without using the arc length parametrization.

The acceleration $\mathbf{a}$ can be decomposed into the tangential and the normal components as follows:

$$\begin{aligned} \mathbf{a}(t) = \frac{d}{dt}\left(\frac{d\mathbf{x}}{dt}\right) = \frac{d}{dt}(\mathbf{v}) = \frac{d}{dt}\left(v\frac{\mathbf{v}}{v}\right) &= \frac{d}{dt}(v\mathbf{T}) \\ &= \dot{v}\mathbf{T} + v\frac{d\mathbf{T}}{dt} \\ &= \dot{v}\mathbf{T} + v\frac{ds}{dt}\frac{d\mathbf{T}}{ds} \\ &= \dot{v}\mathbf{T} + v^2\kappa\mathbf{N}, \end{aligned} \tag{2}$$

where $\dot{v}\mathbf{T}$ is the tangential acceleration, and $v^2\kappa\mathbf{N}$ is the normal acceleration.
Notice that $v = \dot{s}$ and $\kappa\mathbf{N} = \frac{d\mathbf{T}}{ds}$. Also note that the unit vector $\mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{\dot{\mathbf{T}}}{|\dot{\mathbf{T}}|}$. In addition, $\mathbf{N}$ and $\mathbf{v}$ (or $\mathbf{T}$) are orthogonal, i.e., $\mathbf{N}\cdot\mathbf{v} = \mathbf{N}\cdot\mathbf{T} = 0$ and $|\mathbf{N}\times\mathbf{v}| = v$.
Using (2), show that $\kappa(t) = \frac{|\mathbf{v}\times\mathbf{a}|}{v^3}$.

Compute $\kappa(t) = \frac{|\mathbf{v}\times\mathbf{a}|}{v^3}$ of the parameterized curve (1). You may use Matlab symbolics to save your time.

The vector $\kappa\mathbf{N}$ is called curvature normal vector. Its magnitude is the curvature and its direction is the normal direction pointing the center of approximating circle. From (2), the curvature normal is computed as $\kappa\mathbf{N} = (\mathbf{a} - \dot{v}\mathbf{T})/v^2$. Using this formula, compute the curvature normal vector of the curve (1).

Now, we draw the graph by executing the Matlab command:

```
t = 0.1:0.1:8;
plot3(4/3*t.^(3/2),  2*t.*sin(1/2*t),  2*t.*cos(1/2*t));
```

Can you visualize the curvature normal vector in this graph as well? One way is to draw lines from $\mathbf{x}(t)$ to $\mathbf{x}(t) + \kappa(t)\mathbf{T}(t)$. Another way is to draw lines from $\mathbf{x}(t)$ to $\mathbf{x}(t) + \mathbf{T}(t)/\kappa(t)$, which will indicate the center of the circle that approximates the curve at $\mathbf{x}(t)$. Try to visualize these and submit a sketch or a printed copy.

The following is the matlab code to visualize $\mathbf{x}(t) + \kappa(t)\mathbf{T}(t)$.

```
syms t positive

x = [4/3*t^(3/2); 2*t*sin(t/2); 2*t*cos(t/2)]
v = diff(x,t)
v_mag = sqrt(v'*v);
a = diff(v,t)
simplify(sqrt(v'*v))
k = simplify(sqrt(cross(v,a)'*cross(v,a)) / (v_mag^3))

T = simplify(v / v_mag)
kN = simplify( (a - diff(v_mag,t)*T) / v_mag^2)

t = (0.1:0.1:8)';
x = 4/3*t.^(3/2);
y = 2*t.*sin(1/2*t);
z = 2*t.*cos(1/2*t);

kNx =                                                                    -(t-2)./(4*t+4+t.^2)./(2+t)./t.^(1/2);
kNy =  1/2*(-t.^2.*sin(1/2*t)-2*t.*sin(1/2*t)+2*t.*cos(1/2*t)+8*cos(1/2*t)-4*sin(1/2*t))./(t.^3+6*t.^2+12*t+8);
kNz = -1/2*( t.^2.*cos(1/2*t)+2*t.*cos(1/2*t)+2*t.*sin(1/2*t)+8*sin(1/2*t)+4*cos(1/2*t))./(t.^3+6*t.^2+12*t+8);

k = 1/2./(2+t).^2./t.^(1/2).*(t.^3+12*t+4).^(1/2);

figure(1)
plot3([x,x+kNx]',[y,y+kNy]',[z,z+kNz]','g'), hold on;
plot3(x,y,z,'r');
hold off
axis equal
```

Problem 2, Solutions to a linear ODE (stable without oscillation)

Consider an ODE $\dddot{x} + 6\ddot{x} + 11\dot{x} + 6x = 0$. Let $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \ddot{x}$, and $\mathbf{x} = [x_1 \;\; x_2 \;\; x_3]^T$. Compute the matrix $\mathbf{A} \in \mathbb{R}^{3\times 3}$ such that the ODE is equivalent to $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. Using the eigen decomposition of $\mathbf{A}$, compute the solution, i.e, compute $\mathbf{x}(t)$ written as an explicit function of time. Let $x_1(0) = 1, x_2(0) = x_3(0) = 0$. Plot $x_1(t)$ using Matlab. Does $x(t)$ converge to zero?

```
A = [0   1   0
     0   0   1
    -6 -11 -6];

[V,D] = eig(A);
x0 = [1; 0; 0];

x = x0;

for t=0.01:0.01:10
    expDt = [exp(D(1,1)*t)         0              0
                    0        exp(D(2,2)*t)        0
                    0              0      exp(D(3,3)*t)];
    x = [x   V*expDt*inv(V)*x0];
end

plot(0:0.01:10, x(1,:))
```

Problem 3, Solutions to a linear ODE

Consider an ODE $\ddot{x} + 2\dot{x} + 5x = 0$. Let $x_1 = x$, $x_2 = \dot{x}$, and $\mathbf{x} = [x_1 \;\; x_2]^T$. Compute the matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$ such that the ODE is equivalent to $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. Compute the eigenvalues of $\mathbf{A}$. (Since the eigenvalues and vectors are complex numbers, computing explicit solution requires a tedious process that eliminates the complex part. Therefore, let's skip it.) Do you think that $x(t)$ converges to zero as $t \to \infty$?

Problem 4, Solutions to a linear ODE

Consider an ODE $\ddot{x} - 2\dot{x} + 5x = 0$. Let $x_1 = x$, $x_2 = \dot{x}$, and $\mathbf{x} = [x_1 \;\; x_2]^T$. Compute the matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$ such that the ODE is equivalent to $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. Compute the eigenvalues of $\mathbf{A}$. Do you think that $x(t)$ converge to zero as $t \to \infty$?

Problem 5, Characteristic equation

In problems 2,3, and 4, compare the coefficients of the ODEs and the characteristic equations det($\mathbf{A} - \lambda\mathbf{I}$)= 0. Is there a way to compute the eigenvalues of $\mathbf{A}$ (and therefore, be able to tell whether $\mathbf{x}(t)$ converge to zero or diverge to infinity), without constructing $\mathbf{A}$?