

Random Tensors and Planted Cliques

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Abstract

The r -parity tensor of a graph is a generalization of the adjacency matrix, where the tensor's entries denote the parity of the number of edges in subgraphs induced by r distinct vertices. For $r = 2$, it is the adjacency matrix with 1's for edges and -1 's for nonedges. It is well-known that the 2-norm of the adjacency matrix of a random graph is $O(\sqrt{n})$. Here we show that the 2-norm of the r -parity tensor is at most $f(r)\sqrt{n} \log^{O(r)} n$, answering a question of Frieze and Kannan [3] who proved this for $r = 3$. As a consequence, we get a tight connection between the planted clique problem and the problem of finding a vector that approximates the 2-norm of the r -parity tensor of a random graph. Our proof method is based on an inductive application of concentration of measure.

1 Introduction

It is well-known that a random graph $G(n, 1/2)$ almost surely has a clique of size $(2 + o(1)) \log_2 n$ and a simple greedy algorithm finds a clique of size $(1 + o(1)) \log_2 n$. Finding a clique of size even $(1 + \epsilon) \log_2 n$ for some $\epsilon > 0$ in a random graph is a long-standing open problem posed by Karp in 1976 [6] in his classic paper on probabilistic analysis of algorithms.

In the early nineties, a very interesting variant of this question was formulated by Jerrum [5] and by Kucera [7]. Suppose that a clique of size p is planted in a random graph, i.e., a random graph is chosen and all the edges within a subset of p vertices are added to it. Then for what value of p can the planted clique be found efficiently? It is not hard to see that $p > c\sqrt{n \log n}$ suffices since then the vertices of the clique will have larger degrees than the rest of the graph, with high probability [7]. This was improved by Alon et al [1] to $p = \Omega(\sqrt{n})$ using a spectral approach. This was refined by McSherry [8] and considered by Feige and Krauthgamer in the more general semi-random model [2]. For $p \geq 10\sqrt{n}$, the following simple algorithm works: form a matrix with 1's for edges and -1 's for nonedges; find the largest eigenvector of this matrix and read off the top p entries in magnitude; return the set of vertices that have degree at least $3p/4$ within this subset.

The reason this works is the following: the top eigenvector of a symmetric matrix A can be written as

$$\max_{x: \|x\|=1} x^T A x = \max_{x: \|x\|=1} \sum_{ij} A_{ij} x_i x_j$$

maximizing a quadratic polynomial over the unit sphere. The maximum value is the spectral norm or 2-norm of the matrix. For a random matrix with $1, -1$ entries, the spectral norm (largest eigenvalue) is $O(\sqrt{n})$. In fact, as shown by Füredi and Komlós [4, 9], a random matrix with i.i.d. entries of variance at most 1 has the same bound on the spectral norm. On the other hand, after planting a clique of size \sqrt{n} times a sufficient constant factor, the indicator vector of the clique (normalized) achieves a higher norm. Thus the top eigenvector points in the direction of the clique (or very close to it).

Given the numerous applications of eigenvectors (principal components), a well-motivated and natural generalization of this optimization problem to an r -dimensional tensor is the following: given a symmetric

tensor A with entries $A_{k_1 k_2 \dots k_r}$, find

$$\|A\|_2 = \max_{x: \|x\|=1} A(x, \dots, x),$$

where

$$A(x^{(1)}, \dots, x^{(r)}) = \sum_{i_1 i_2 \dots i_r} A_{i_1 i_2 \dots i_r} x_{i_1}^{(1)} x_{i_2}^{(2)} \dots x_{i_r}^{(r)}.$$

The maximum value is the spectral norm or 2-norm of the tensor. The complexity of this problem is open for any $r > 2$, assuming the entries with repeated indices are zeros.

A beautiful application of this problem was given recently by Frieze and Kannan [3]. They defined the following tensor associated with an undirected graph $G = (V, E)$:

$$A_{ijk} = E_{ij} E_{jk} E_{ki}$$

where E_{ij} is 1 if $ij \in E$ and -1 otherwise, i.e., A_{ijk} is the parity of the number of edges between i, j, k present in G . They proved that for the random graph $G_{n,1/2}$, the 2-norm of the random tensor A is $\tilde{O}(\sqrt{n})$, i.e.,

$$\sup_{x: \|x\|=1} \sum_{i,j,k} A_{ijk} x_i x_j x_k \leq C \sqrt{n} \log^c n$$

where c, C are absolute constants. This implied that if such a maximizing vector x could be found (or approximated), then we could find planted cliques of size as small as $n^{1/3}$ times polylogarithmic factors in polynomial time, improving substantially on the long-standing threshold of $\Omega(\sqrt{n})$.

Frieze and Kannan ask the natural question of whether this connection can be further strengthened by going to r -dimensional tensors for $r > 3$. The tensor itself has a nice generalization. For a given graph $G = (V, E)$ the r -parity tensor is defined as follows. Entries with repeated indices are set to zero; any other entry is the parity of the number of edges in the subgraph induced by the subset of vertices corresponding to the entry, i.e.,

$$A_{k_1, \dots, k_r} = \prod_{1 \leq i < j \leq r} E_{k_i k_j}.$$

Frieze and Kannan's proof for $r = 3$ is combinatorial (as is the proof by Füredi and Komlós for $r = 2$), based on counting the number of subgraphs of a certain type. It is not clear how to extend this proof.

Here we prove a nearly optimal bound on the spectral norm of this random tensor for any r . This substantially strengthens the connection between the planted clique problem and the tensor norm problem. Our proof is based on a concentration of measure approach. In fact, we first reprove the result for $r = 3$ using this approach and then generalize it to tensors of arbitrary dimension. We show that the norm of the subgraph parity tensor of a random graph is at most $f(r)\tilde{O}(\sqrt{n})$ whp. More precisely, our main theorem is the following.

Theorem 1. *There is a constant C_1 such that with probability at least $1 - n^{-1}$ the norm of the r -dimensional subgraph parity tensor $A : [n]^r \rightarrow \{-1, 1\}$ for the random graph $G_{n,1/2}$ is bounded by*

$$\|A\|_2 \leq C_1^r r^{(5r-1)/2} \sqrt{n} \log^{(3r-1)/2} n.$$

The main challenge to the proof is the fact that the entries of the tensor A are not independent. Bounding the norm of the tensor where every entry is independently 1 or -1 with probability $1/2$ is substantially easier via a combination of an ϵ -net and a Hoeffding bound. In more detail, we approximate the unit ball with a finite (exponential) set of vectors. For each vector x in the discretization, the Hoeffding inequality gives an exponential tail bound on $A(x, \dots, x)$. A union bound over all points in the discretization then completes the proof. For the parity tensor, however, the Hoeffding bound does not apply as the entries are not independent. Moreover, all the $\binom{n}{r}$ entries of the tensor are fixed by just the $\binom{n}{2}$ edges of the graph. In spite of this heavy inter-dependence, it turns out that $A(x, \dots, x)$ does concentrate. Our proof is inductive and bounds the norms of vectors encountered in a certain decomposition of the tensor polynomial.

Using Theorem 1, we can show that if the norm problem can be solved for tensors of dimension r , one can find planted cliques of size as low as $Cn^{1/r}\text{poly}(r, \log n)$. While the norm of the parity tensor for a random graph remains bounded, when a clique of size p is planted, the norm becomes at least $p^{r/2}$ (using the indicator vector of the clique). Therefore, p only needs to be a little larger than $n^{1/r}$ in order for the clique to become the dominant term in the maximization of $A(x, \dots, x)$. More precisely, we have the following theorem.

Theorem 2. *Let G be random graph $G_{n,1/2}$ with a planted clique of size p , and let A be the r -parity tensor for G . For $\alpha \leq 1$, let $T(n, r)$ be the time to compute a vector x such that $A(x, \dots, x) \geq \alpha^r \|A\|_2$ whp. Then, for p such that*

$$n \geq p > C_0 \alpha^{-2} r^5 n^{1/r} \log^3 n,$$

the planted clique can be recovered with high probability in time $T(n, r) + \text{poly}(n)$, where C_0 is a fixed constant.

On one hand, this highlights the benefits of finding an efficient (approximation) algorithm for the tensor problem. On the other, given the lack of progress on the clique problem, this is perhaps evidence of the hardness of the tensor maximization problem even for a natural class of random tensors. For example, if finding a clique of size $\tilde{O}(n^{1/2-\epsilon})$ is hard, then by setting $\alpha = n^{1/2r+\epsilon/2-1/4}$ we see that even a certain polynomial approximation to the norm of the parity tensor is hard to achieve.

Corollary 3. *Let G be random graph $G_{n,1/2}$ with a planted clique of size p , and let A be the r -parity tensor for G . Let $\epsilon > 0$ be a small constant and let $T(n, r)$ be the time to compute a vector x such that $A(x, \dots, x) \geq n^{1/2+r\epsilon/2-r/4} \|A\|_2$. Then, for*

$$p \geq C_0 r^5 n^{\frac{1}{2}-\epsilon} \log^3 n,$$

the planted clique can be recovered with high probability in time $T(n, r) + \text{poly}(n)$, where C_0 is a fixed constant.

1.1 Overview of analysis

The majority of the paper is concerned with proving Theorem 1. In Section 2.1, we first reduce the problem of bounding $A(\cdot)$ over the unit ball to bounding it over a discrete set of vectors that have the same value in every non-zero coordinate. In Section 2.2, we further reduce the problem to bounding the norm of an off-diagonal block of A , using a method of Frieze and Kannan. This enables us to assume that if (k_1, \dots, k_r) is a valid index, then the random variables E_{k_i, k_j} used to compute A_{i_1, \dots, i_r} are independent. In Section 2.3, we prove a large deviation inequality (Lemma 6) that allows us to bound norms of vectors encountered in a certain decomposition of the tensor polynomial. This inequality gives us a considerably sharper bound than the Hoeffding or McDiarmid inequalities in our context. We then apply this lemma to bound $\|A\|_2$ for $r = 3$ as a warm-up and then give the proof for general r in Section 3.

In Section 4 we prove Theorem 2. We first show that any vector x that comes close to maximizing $A(\cdot)$ must be close to the indicator vector of the clique (Lemma 4). Finally, we show that given such a vector it is possible to recover the clique (Lemma 14).

2 Preliminaries

2.1 Discretization

The analysis of $A(x, \dots, x)$ is greatly simplified when x is proportional to some indicator vector. Fortunately, analyzing these vectors is sufficient, as any vector can be approximated as a linear combination of relatively few indicator vectors.

For any vector x , we define $x^{(+)}$ to be vector such that $x_i^{(+)} = x_i$ if $x_i > 0$ and $x_i^{(+)} = 0$ otherwise. Similarly, let $x_i^{(-)} = x_i$ if $x_i < 0$ and $x_i^{(-)} = 0$ otherwise. For a set $S \subseteq [n]$, let χ^S be the indicator vector for S , where the i th entry is 1 if $i \in S$ and 0 otherwise.

Definition 1 (Indicator Decomposition). For a unit vector x , define the sets S_1, \dots and T_1, \dots through the recurrences

$$S_j = \left\{ i \in [n] : (x^{(+)} - \sum_{k=1}^{j-1} 2^{-k} \chi^{S_k})_i > 2^{-j} \right\}.$$

and

$$T_j = \left\{ i \in [n] : (x^{(-)} - \sum_{k=1}^{j-1} 2^{-k} \chi^{S_k})_i < -2^{-j} \right\}.$$

Let $y_0(x) = 0$. For $j \geq 1$, let $y^{(j)}(x) = 2^{-j} \chi^{S_j}$ and let $y^{(-j)}(x) = -2^{-j} \chi^{T_j}$. We call the set $\{y^{(j)}(x)\}_{j=-\infty}^{\infty}$ the *indicator decomposition* of x .

Clearly,

$$\|y^{(i)}(x)\| \leq \max\{\|x^{(+)}\|, \|x^{(-)}\|\} \leq 1.$$

and

$$\left\| x - \sum_{j=-N}^N y^{(j)}(x) \right\| \leq \sqrt{n} 2^{-N}.$$

We use this decomposition to prove the following theorem.

Lemma 4. *Let*

$$U = \{k|S|^{-1/2} \chi^S : S \subseteq [n], k \in \{-1, 1\}\}.$$

For any tensor A over $[n]^r$ where $\|A\|_{\infty} \leq 1$

$$\max_{x^{(1)}, \dots, x^{(r)} \in B(0,1)} A(x^{(1)}, \dots, x^{(r)}) \leq (2 \lceil r \log_2 n \rceil)^r \max_{x^{(1)}, \dots, x^{(r)} \in U} A(x^{(1)}, \dots, x^{(r)})$$

Proof. Consider a fixed set of vectors $x^{(1)}, \dots, x^{(r)}$ and let $N = \lceil r \log_2 n \rceil$. For each i , let

$$x^{\hat{(i)}} = \sum_{j=-N}^N y^{(j)}(x^{(i)}).$$

We first show that replacing $x^{(i)}$ with $x^{\hat{(i)}}$ gives a good approximation to $A(x^{(1)}, \dots, x^{(r)})$. Letting ϵ be the maximum difference between an $x^{(i)}$ and its approximation, we have that

$$\max_{i \in [r]} \|x^{(i)} - x^{\hat{(i)}}\| = \epsilon \leq \frac{n^{r/2}}{2^r}$$

Because of the multilinear form of $A(\cdot)$ we have

$$\begin{aligned} |A(x^{(1)}, \dots, x^{(r)}) - A(x^{\hat{(1)}}, \dots, x^{\hat{(r)}})| &\leq \sum_{i=1}^r \epsilon^i r^i \|A\| \\ &\leq \frac{\epsilon^r}{1 - \epsilon^r} \|A\| \\ &\leq n^{-r/2} \|A\| \\ &\leq 1. \end{aligned}$$

Next, we bound $A(x^{\hat{(1)}}, \dots, x^{\hat{(r)}})$. For convenience, let $Y^{(i)} = \cup_{j=-N}^N y^{(j)}(x^{(i)})$. Then using the multilinear form of $A(\cdot)$ and bounding the sum by its maximum term, we have

$$\begin{aligned} A(x^{\hat{(1)}}, \dots, x^{\hat{(r)}}) &\leq (2N)^r \max_{v^{(1)} \in Y^{(1)}, \dots, v^{(r)} \in Y^{(r)}} A(v^{(1)}, \dots, v^{(r)}) \\ &\leq (2N)^r \max_{v^{(1)}, \dots, v^{(r)} \in U} A(v^{(1)}, \dots, v^{(r)}). \end{aligned}$$

□

2.2 Sufficiency of off-diagonal blocks

Analysis of $A(x^{(1)}, \dots, x^{(r)})$ is complicated by the fact that all terms with repeated indices are zero. Off-diagonal blocks of A are easier to analyze because no such terms exist. Thankfully, as Frieze and Kannan [3] have shown, analyzing these off-diagonal blocks suffices. Here we generalize their proof to $r > 3$.

For a collection $\{V_1, V_2, \dots, V_r\}$ of subsets of $[n]$, we define

$$A|_{V_1 \times \dots \times V_r}(x^{(1)}, \dots, x^{(r)}) = \sum_{k_1 \in V_1, \dots, k_r \in V_r} A_{k_1 \dots k_r} x_{i_1}^{(1)} x_{i_2}^{(2)} \dots x_{i_r}^{(r)}$$

Lemma 5. *Let P be the class of partitions of $[n]$ into r equally sized sets V_1, \dots, V_r (assume wlog that r divides n). Let $V = V_1 \times \dots \times V_r$. Let A be a random tensor over $[n]^r$ where each entry is in $[-1, 1]$ and let $R \subseteq B(0, 1)$. If for every fixed $(V_1, \dots, V_r) \in P$, it holds that*

$$\Pr\left[\max_{x^{(1)}, \dots, x^{(r)} \in R} A|_V(x^{(1)}, \dots, x^{(r)}) \geq f(n)\right] \leq \delta,$$

then

$$\Pr\left[\max_{x^{(1)}, \dots, x^{(r)} \in R} A(x^{(1)}, \dots, x^{(r)}) \geq 2r^r f(n)\right] \leq \frac{\delta n^{r/2}}{f(n)},$$

Proof of Lemma 5. Each r -tuple appears in an equal number of partitions and this number is slightly more than a r^{-r} fraction of the total. Therefore,

$$\begin{aligned} \left|A(x^{(1)}, \dots, A(x^{(r)})\right| &\leq \frac{r^r}{|P|} \left| \sum_{\{V_1, \dots, V_r\} \in P} A|_V(x^{(1)}, \dots, A(x^{(r)})) \right| \\ &\leq \frac{r^r}{|P|} \sum_{\{V_1, \dots, V_r\} \in P} \left|A|_V(x^{(1)}, \dots, A(x^{(r)}))\right| \end{aligned}$$

We say that a partition $\{V_1, \dots, V_r\}$ is good if

$$\max_{x^{(1)}, \dots, x^{(r)} \in R} A|_V(x^{(1)}, \dots, x^{(r)}) < f(n).$$

Let the good partitions be denoted by G and let $\bar{G} = P \setminus G$. Although the f upper bound does not hold for partitions in \bar{G} , the trivial upper bound of $n^{r/2}$ does (recall that every entry in the tensor is in the range $[-1, 1]$ and $R \subseteq B(0, 1)$). Therefore

$$\left|A(x^{(1)}, \dots, A(x^{(r)})\right| \leq r^r \left(\frac{|G|}{|P|} f + \frac{|\bar{G}|}{|P|} n^{r/2} \right).$$

Since $E[|G|/|P|] = \delta$ by hypothesis, Markov's inequality gives

$$\Pr\left[\frac{|G|}{|P|} n^{r/2} > f\right] \leq \frac{\delta n^{r/2}}{f}$$

and thus proves the result. \square

2.3 A concentration bound

The following concentration bound is a key tool in our proof of Theorem 1. We apply it for $t = \tilde{O}(N)$.

Lemma 6. *Let $\{u^{(i)}\}_{i=1}^N$ and $\{v^{(i)}\}_{i=1}^N$ be collections of vectors of dimension N' where each entry of $u^{(i)}$ is 1 or -1 with probability $1/2$ and $\|v^{(i)}\|_2 \leq 1$. Then for any $t \geq 1$,*

$$\Pr\left[\sum_{i=1}^N (u^{(i)} \cdot v^{(i)})^2 \geq t\right] \leq e^{-t/18} (4\sqrt{e\pi})^N.$$

Before giving the proof, we note that this lemma is stronger than what a naive application of standard theorems would yield for $t = \tilde{O}(N)$. For instance, one might treat each $(u^{(i)} \cdot v^{(i)})^2$ as an independent random variable and apply a Hoeffding bound. The quantity $(u^{(i)} \cdot v^{(i)})^2$ can vary by as much as N' , however, so the bound would be roughly $\exp(-ct^2/NN'^2)$ for some constant c . Similarly, treating each $u_j^{(i)}$ as an independent random variable and applying McDiarmid's inequality, we find that every $u_j^{(i)}$ can affect the sum by as much as 1 (simultaneously). For instance suppose that every $v_j^{(i)} = 1/\sqrt{N'}$ and every $u_j^{(i)} = 1$. Then flipping $u_j^{(i)}$ would have an effect of $|N' - ((N' - 2)/\sqrt{N'})^2| \approx 4$, so the bound would be roughly $\exp(-ct^2/NN')$ for some constant c .

Proof of Lemma 6. Observe that $\sqrt{\sum_{i=1}^N (u^{(i)} \cdot v^{(i)})^2}$ is the length of the vector whose i th coordinate is $u^{(i)} \cdot v^{(i)}$. Therefore, this is also equivalent to the maximum projection of this vector onto a unit vector:

$$\sqrt{\sum_{i=1}^N (u^{(i)} \cdot v^{(i)})^2} = \max_{y \in B(0,1)} \sum_{i=1}^N \sum_{j=1}^{N'} y_i u_j^{(i)} v_j^{(i)}.$$

We will use an ϵ -net to approximate the unit ball and give an upper bound for this quantity. Let \mathcal{L} be the lattice $\left(\frac{1}{2\sqrt{N}}\mathbb{Z}\right)^N$.

Claim 7. For any vector x ,

$$\|x\|_2 \leq 2 \max_{y \in \mathcal{L} \cap B(0,3/2)} y \cdot x.$$

Thus,

$$\sqrt{\sum_{i=1}^N (u^{(i)} \cdot v^{(i)})^2} \leq 2 \max_{y \in \mathcal{L} \cap B(0,3/2)} \sum_{i=1}^N y_i \sum_{j=1}^{N'} u_j^{(i)} v_j^{(i)}.$$

Consider a fixed $y \in \mathcal{L} \cap B(0,3/2)$. Each $u_j^{(i)}$ is 1 or -1 with equal probability, so the expectation for each term is zero. The difference between the upper and lower bounds for a term is

$$2|2y_j u_j^{(i)} v(i)_j| = 4|y_j v(i)_j|$$

Therefore,

$$16 \sum_{i=1}^N \sum_{j=1}^{N'} (y_i u_j^{(i)} v(i)_j)^2 \leq 16 \sum_{i=1}^N y_i^2 \sum_{j=1}^{N'} (v(i)_j)^2 = 36.$$

Applying the Hoeffding bound gives that

$$\Pr\left[\sum_{i=1}^N (u^{(i)} \cdot v^{(i)})^2 \geq t\right] \leq \Pr\left[2 \sum_{i=1}^N y_i \sum_{j=1}^{N'} u_j^{(i)} v(i)_j \geq \sqrt{t}\right] \leq e^{-t/18}.$$

The result follows by taking a union bound over $\mathcal{L} \cap B(0,3/2)$, whose cardinality is bounded according to Claim 8. \square

Claim 8. The number of lattice points in $\mathcal{L} \cap B(0,3/2)$ is at most $(4\sqrt{e\pi})^N$

Proof of Claim 8. Consider the set of hypercubes where each cube is centered on a distinct point in $\mathcal{L} \cap B(0,3/2)$ and each has side length of $(2\sqrt{n})^{-1}$. These cubes are disjoint and their union contains the ball

$B(0, 3/2)$. Their union is also contained in the ball $B(0, 2)$. Thus,

$$\begin{aligned} |\mathcal{L} \cap B(0, 3/2)| &\leq \frac{\text{Vol}(B(0, 2))}{(2\sqrt{N})^{-N}} \\ &\leq \frac{\pi^{N/2} 2^N}{\Gamma(N/2 + 1)} 2^N N^{N/2} \\ &\leq (4\sqrt{e\pi})^N. \end{aligned}$$

□

Proof of Claim 7. Without loss of generality, we assume that x is a unit vector. Let y be the closest point to x in the lattice. In each coordinate i , we have $|x_i - y_i| \leq (4\sqrt{n})^{-1}$, so overall $\|x - y\| \leq 1/4$.

Letting θ be the angle between x and y , we have

$$\frac{x \cdot y}{\|x\|\|y\|} = \cos \theta = \sqrt{1 - \sin^2 \theta} \geq \left(1 - \frac{\|x - y\|^2}{\max\{\|x\|^2, \|y\|^2\}}\right)^{1/2} \geq \sqrt{\frac{15}{16}}.$$

Therefore,

$$x \cdot y \geq \|y\| \sqrt{\frac{15}{16}} \geq \frac{3}{4} \sqrt{\frac{15}{16}} \geq \frac{1}{2}.$$

□

3 A bound on the norm of the parity tensor

In this section, we prove Theorem 1. First, however, we consider the somewhat more transparent case of $r = 3$ using the same proof technique.

3.1 Warm-up: third order tensors

For $r = 3$ the tensor A is defined as follows:

$$A_{k_1 k_2 k_3} = E_{k_1 k_2} E_{k_2 k_3} E_{k_1 k_3}.$$

Theorem 9. *There is a constant C_1 such that with probability $1 - n^{-1}$*

$$\|A\| \leq C_1 \sqrt{n} \log^4 n.$$

Proof. Let V_1, V_2, V_3 be a partition of the n vertices and let $V = V_1 \times V_2 \times V_3$. The bulk of the proof consists of the following lemma.

Lemma 10. *There is some constant C_3 such that*

$$\max_{x^{(1)}, x^{(2)}, x^{(3)} \in U} A|_V(x^{(1)}, x^{(2)}, x^{(3)}) \leq C_3 \sqrt{n} \log n$$

with probability $1 - n^{-7}$.

If this bound holds, then Lemma 4 then implies that there is some C_2 such that

$$\max_{x^{(1)}, x^{(2)}, x^{(3)} \in B(0, 1)} A|_V(x^{(1)}, x^{(2)}, x^{(3)}) \leq C_2 \sqrt{n} \log^4 n.$$

And finally, Lemma 5 implies that for some constant C_1

$$\max_{x^{(1)}, x^{(2)}, x^{(3)} \in B(0, 1)} A(x^{(1)}, x^{(2)}, x^{(3)}) \leq C_1 \sqrt{n} \log^4 n$$

with probability $1 - n^{-1}$ for some constant C_1 . □

Proof of Lemma 10. Define

$$U_k = \{x \in U : |\text{supp}(x)| = k\} \quad (1)$$

and consider a fixed $n \geq n_1 \geq n_2 \geq n_3 \geq 1$. We will show that

$$\max_{(x^{(1)}, x^{(2)}, x^{(3)}) \in U_{n_1} \times U_{n_2} \times U_{n_3}} A|_V(x^{(1)}, x^{(2)}, x^{(3)}) \leq C_3 \sqrt{n} \log n$$

with probability n^{-10} for some constant C_3 . Taking a union bound over the n^3 choices of n_1, n_2, n_3 then proves the lemma.

We bound the cubic form as

$$\begin{aligned} & \max_{(x^{(1)}, x^{(2)}, x^{(3)}) \in U_{n_1} \times U_{n_2} \times U_{n_3}} A|_V(x^{(1)}, x^{(2)}, x^{(3)}) \\ &= \max_{(x^{(1)}, x^{(2)}, x^{(3)}) \in U_{n_1} \times U_{n_2} \times U_{n_3}} \sum_{k_1 \in V_1, k_2 \in V_2, k_3 \in V_3} A_{k_1 k_2 k_3} x_{k_1}^{(1)} x_{k_2}^{(2)} x_{k_3}^{(3)} \\ &\leq \max_{(x^{(2)}, x^{(3)}) \in U_{n_2} \times U_{n_3}} \sqrt{\sum_{k_1 \in V_1} \left(\sum_{k_2 \in V_2, k_3 \in V_3} A_{k_1 k_2 k_3} x_{k_2}^{(2)} x_{k_3}^{(3)} \right)^2} \\ &= \max_{(x^{(2)}, x^{(3)}) \in U_{n_2} \times U_{n_3}} \sqrt{\sum_{k_1 \in V_1} \left(\sum_{k_2 \in V_2} E_{k_1 k_2} x_{k_2}^{(2)} \sum_{k_3 \in V_3} E_{k_2 k_3} x_{k_3}^{(3)} E_{k_1 k_3} \right)^2}. \end{aligned}$$

Note that each of the inner sums (over k_2 and k_3) are the dot product of a random $-1, 1$ vector (the $E_{k_1 k_2}$ and $E_{k_2 k_3}$ terms) and another vector. Our strategy will be to bound the norm of this other vector and apply Lemma 6.

In more detail, we view the expression inside the square root a

$$\sum_{k_1 \in V_1} \left(\underbrace{\sum_{k_2 \in V_2} \underbrace{\overbrace{E_{k_1 k_2} x_{k_2}^{(2)} \sum_{k_3 \in V_3} \underbrace{\overbrace{E_{k_2 k_3} x_{k_3}^{(3)} E_{k_1 k_3}}^{v^{(k_1 k_2)}(x^{(3)})_{k_3}}} \right)}_{u^{(k_2)} \cdot v^{(k_1 k_2)}(x^{(3)})} x_{k_2}^{(2)}}^{u^{(k_1)} \cdot v^{(k_1)}(x^{(2), x^{(3)})_{k_2}}} \right)^2 \quad (2)$$

where $u_{k_3}^{(k_2)} = E_{k_2 k_3}$ and $u_{k_2}^{(k_1)} = E_{k_1 k_2}$, while

$$v^{(k_1 k_2)}(x^{(3)})_{k_3} = x_{k_3}^{(3)} E_{k_1 k_3}$$

and

$$v^{(k_1)}(x^{(2)}, x^{(3)})_{k_2} = x_{k_2}^{(2)} (u^{(k_2)} \cdot v^{(k_1 k_2)}(x^{(3)})).$$

Clearly, the u 's play the role of the random vectors and we will bound the norms of the v 's in the application of Lemma 6.

To apply Lemma 6 with k_1 being the index i , $u_{k_2}^{k_1} = E_{k_1 k_2}$ above, we need a bound for every $k_1 \in V_1$ on

the norm of $v^{(k_1)}(x^{(2)}, x^{(3)})$. We argue

$$\begin{aligned} & \sum_{k_2} \left(x_{k_2}^{(2)} \sum_{k_3 \in V_3} E_{k_2 k_3} x_{k_3}^{(3)} E_{k_1 k_3} \right)^2 \\ & \leq \max_{k_1 \in V_1} \max_{x^{(2)} \in U_{n_2}} \max_{x^{(3)} \in U_{n_3}} \frac{1}{n_2} \sum_{k_2 \in \text{supp}(x^{(2)})} \left(\sum_{k_3} E_{k_2 k_3} x_{k_3}^{(3)} E_{k_1 k_3} \right)^2 \\ & = F_1^2 \end{aligned}$$

Here we used the fact that $\|x^{(2)}\|_\infty \leq n_2^{-1/2}$. Note that F_1 is a function of the random variables $\{E_{ij}\}$ only.

To bound F_1 , we observe that we can apply Lemma 6 to the expression being maximized above, i.e.,

$$\sum_{k_2} \left(\sum_{k_3} E_{k_2 k_3} \left(x_{k_3}^{(3)} E_{k_1 k_3} \right) \right)^2$$

over the index k_2 , with $u_{k_3}^{k_2} = E_{k_2 k_3}$. Now we need a bound, for every k_2 and k_1 on the norm of the vector $v^{(k_1 k_2)}(x^{(3)})$. We argue

$$\begin{aligned} \sum_{k_3} \left(x_{k_3}^{(3)} E_{k_1 k_3} \right)^2 & \leq \|x^{(3)}\|_\infty^2 \sum_{k_3} E_{k_1 k_3}^2 \\ & \leq 1. \end{aligned}$$

Applying Lemma 6 for a fixed $k_1, x^{(2)}$ and $x^{(3)}$ implies

$$\frac{1}{n_2} \sum_{k_2 \in \text{supp}(x^{(2)})} \left(\sum_{k_3} E_{k_2 k_3} x_{k_3}^{(3)} E_{k_1 k_3} \right)^2 > C_3 \log n$$

with probability at most

$$\exp\left(-\frac{C_3 n_2 \log n}{18}\right) (4\sqrt{e\pi})^{n_2}.$$

Taking a union bound over the $|V_1| \leq n$ choices of k_1 , and the at most $n^{n_2} n^{n_3}$ choices for $x^{(2)}$ and $x^{(3)}$, we show that

$$\Pr[F_1^2 > C_3 \log n] \leq \exp\left(-\frac{C_3 n_2 \log n}{18}\right) (4\sqrt{e\pi})^{n_2} n n^{n_2} n^{n_3}.$$

This probability is at most $n^{-10}/2$ for a large enough constant C_3 .

Thus, for a fixed $x^{(2)}$ and $x^{(3)}$, we can apply Lemma 6 to Eqn. 2 with $F_1^2 = C_3 \log n$ to get:

$$\sum_{k_1 \in V_1} \left(\sum_{k_2 \in V_2} E_{k_1 k_2} \left(x_{k_2}^{(2)} \sum_{k_3 \in V_3} E_{k_2 k_3} x_{k_3}^{(3)} E_{k_1 k_3} \right) \right)^2 > F_1^2 C_3 n \log n$$

with probability at most $\exp(-C_3 n \log n / 18) (4\sqrt{e\pi})^n$. Taking a union bound over the at most $n^{n_2} n^{n_3}$ choices for $x^{(2)}$ and $x^{(3)}$, the bound holds with probability

$$\exp(-C_3 n \log n / 18) (4\sqrt{e\pi})^n n^{n_2} n^{n_3} \leq n^{-10}/2$$

for large enough constant C_3 .

Thus, we can bound the squared norm:

$$\begin{aligned}
& \max_{(x^{(1)}, x^{(2)}, x^{(3)}) \in U_{n_1} \times U_{n_2} \times U_{n_3}} A|_V(x^{(1)}, x^{(2)}, x^{(3)})^2 \\
& \leq \sum_{k_1 \in V_1} \left(\sum_{k_2 \in V_2} E_{k_1 k_2} \left(x_{k_2}^{(2)} \sum_{k_3 \in V_3} E_{k_2 k_3} x_{k_3}^{(3)} E_{k_1 k_3} \right) \right)^2 \\
& \leq C_3^2 n_1 \log^2 n
\end{aligned}$$

with probability $1 - n^{-10}$. □

3.2 Higher order tensors

Let the random tensor A be defined as follows.

$$A_{k_1, \dots, k_r} = \prod_{1 \leq i < j \leq r} E_{k_i k_j}$$

where E is an $n \times n$ matrix where each off-diagonal entry is -1 or 1 with probability $1/2$ and every diagonal entry is 1 .

For most of this section, we will consider only a single off-diagonal cube of A . That is, we index over $V_1 \times \dots \times V_r$ where V_i are an equal partition of $[n]$. We denote this block by $A|_V$. When k_i is used as an index, it is implied that $k_i \in V_i$.

The bulk of the proof consists of the following lemma.

Lemma 11. *There is some constant C_3 such that*

$$\max_{x^{(1)}, \dots, x^{(r)} \in U} A|_V(x^{(1)}, \dots, x^{(r)})^2 \leq n(C_3 r \log n)^{r-1}$$

with probability $1 - n^{-9r}$.

The key idea is that Lemma 6 can be applied repeatedly to collections of u 's and v 's in a way analogous to Eqn. 2. Each sum over k_r, \dots, k_2 contributes a $C_3 r \log n$ factor and the final sum over k_1 contributes the factor of n .

If the bound holds, then Lemma 4 implies that there is some C_2 such that

$$\max_{x^{(1)}, x^{(2)}, x^{(3)} \in B(0,1)} A|_V(x^{(1)}, x^{(2)}, x^{(3)})^2 \leq C_2^r r^{2r+r-1} n \log^{2r+(r-1)} n.$$

And finally, Lemma 5 implies that for some constant C_1

$$\begin{aligned}
\max_{x^{(1)}, x^{(2)}, x^{(3)} \in B(0,1)} A(x^{(1)}, x^{(2)}, x^{(3)}) & \leq C_1^r r^{2r+2r+(r-1)} n \log^{2r+r-1} n \\
& = C_1^r r^{5r-1} n \log^{3r-1} n.
\end{aligned}$$

with probability $1 - n^{-1}$ for some constant C_1 .

Proof of Lemma 11. We define the set U_k as in Eqn. 1. It suffices to show that the bound

$$\max_{(x^{(1)}, \dots, x^{(r)}) \in U_{n_1} \times \dots \times U_{n_r}} A|_V(x^{(1)}, \dots, x^{(r)})^2 \leq n(C_3 r \log n)^{r-1}$$

holds with probability $1 - n^{-10r}$ for some constant C_3 , since we may then take a union bound over the n^r choices of $n \geq n_1 \geq \dots \geq n_r \geq 1$.

For convenience of notation, we define a family of tensors as follows

$$B_{k_{\ell+1}, \dots, k_r}^{(k_1, \dots, k_\ell)} = \prod_{i, j: i, \ell < j} E_{k_i k_j} \quad (3)$$

where the superscript indexes the family of tensors and the subscript indexes the entries. Note that for every $k_1, \dots, k_r \in V_1 \times \dots \times V_r$, we have $B^{(k_1, \dots, k_r)} = 1$, since the product is empty.

Note that the tensor $B^{(k_1, \dots, k_\ell)}$ depends only a subset of E . In particular, any such tensor of order $r - \ell$ will depend only on the blocks of E

$$F_\ell = \{E|_{V_i \times V_j} : i, \ell < j\}.$$

Clearly, $F_r = \emptyset$, F_1 contains all blocks, and $F_\ell \setminus F_{\ell+1} = \{E|_{V_i \times V_{\ell+1}} : i \leq \ell\}$.

We bound the r th degree form as

$$\begin{aligned} & \max_{x^{(1)}, \dots, x^{(r)} \in U_{n_1} \times \dots \times U_{n_r}} A|_V(x^{(1)}, \dots, x^{(r)}) \\ &= \max_{x^{(1)}, \dots, x^{(r)} \in U_{n_1} \times \dots \times U_{n_r}} \sum_{k_1 \in V_1} x_{k_1}^{(1)} B^{(k_1)}(x^{(2)}, \dots, x^{(r)}) \\ &\leq \max_{x^{(2)}, \dots, x^{(r)} \in U_{n_2} \times \dots \times U_{n_r}} \sqrt{\sum_{k_1 \in V_1} B^{(k_1)}(x^{(2)}, \dots, x^{(r)})^2}. \end{aligned} \quad (4)$$

Observe that for a general ℓ ,

$$B^{(k_1, \dots, k_\ell)}(x^{(\ell+1)}, \dots, x^{(r)}) = \sum_{k_{\ell+1} \in V_{\ell+1}} E_{k_\ell k_{\ell+1}} v^{(k_1, \dots, k_\ell)}(x^{(\ell+1)}, \dots, x^{(r)})_{k_{\ell+1}}, \quad (5)$$

where

$$v^{(k_1, \dots, k_\ell)}(x^{(\ell+1)}, \dots, x^{(r)})_{k_{\ell+1}} = x_{k_{\ell+1}}^{(\ell+1)} B^{(k_1, \dots, k_{\ell+1})}(x^{(\ell+2)}, \dots, x^{(r)}) \prod_{i < \ell} E_{k_i k_{\ell+1}}. \quad (6)$$

It will be convenient to think of $B^{(k_1, \dots, k_\ell)}(x^{(\ell+1)}, \dots, x^{(r)})$ as the dot product of a random vector $u^{(k_\ell)}$, where $u_{k_{\ell+1}}^{(k_\ell)} = E_{k_\ell k_{\ell+1}}$ and $v^{(k_1, \dots, k_\ell)}(x^{(\ell+1)}, \dots, x^{(r)})_{k_{\ell+1}}$, so that

$$B^{(k_1, \dots, k_\ell)}(x^{(\ell+1)}, \dots, x^{(r)}) = u^{(k_\ell)} \cdot v^{(k_1, \dots, k_\ell)}(x^{(\ell+1)}, \dots, x^{(r)}). \quad (7)$$

The sum over $k_1 \in V_1$ from Eqn. 4 can therefore be expanded as

$$\sum_{k_1 \in V_1} B^{(k_1)}(x^{(2)}, \dots, x^{(r)})^2 = \sum_{k_1 \in V_1} \left(u^{(k_1)} \cdot v^{(k_1)}(x^{(2)}, \dots, x^{(r)}) \right)^2.$$

Our goal is to bound $\|v^{(k_1)}(x^{(2)}, \dots, x^{(r)})\|$ and apply Lemma 6. Notice that for general ℓ

$$\begin{aligned} & \left\| v^{(k_1, \dots, k_\ell)}(x^{(\ell+1)}, \dots, x^{(r)}) \right\|_2^2 \\ &= \frac{1}{n^{\ell+1}} \sum_{k_{\ell+1} \in \text{supp}(x^{(\ell+1)})} B^{(k_1, \dots, k_{\ell+1})}(x^{(\ell+2)}, \dots, x^{(r)})^2 \\ &\leq \max_{k_1, \dots, k_\ell} \max_{x^{(\ell+1)} \in U_{n_{\ell+1}} \dots x^{(r)} \in U_{n_r}} \frac{1}{n^{\ell+1}} \sum_{k_{\ell+1} \in \text{supp}(x^{(\ell+1)})} B^{(k_1, \dots, k_{\ell+1})}(x^{(\ell+2)}, \dots, x^{(r)})^2 = f_\ell^2. \end{aligned} \quad (8)$$

Note that the quantity f_ℓ (define above) depends only on the blocks $F_{\ell+1}$.

The following claims will establish a probabilistic bound on f_1 .

Claim 12. *The quantity*

$$f_{r-1} = 1.$$

Proof. Trivially, every $B^{(k_1, \dots, k_r)}()^2 = 1$. Therefore, for every subset $S_r \subseteq V_r$ such that $|S_r| = n_r$

$$\frac{1}{n_r} \sum_{k_r \in S_r} B^{(k_1, \dots, k_r)}()^2 = 1.$$

□

Claim 13. *There is a constant C_3 such that for any $\ell \in 1 \dots r - 2$*

$$\Pr[f_\ell^2 > C_3 r f_{\ell+1}^2 \log n] \leq n^{-12r}.$$

We postpone the proof of Claim 13 and argue that by induction we have that

$$f_1^2 \leq (C_3 r \log n)^{r-2}$$

with probability $1 - n^{-12r} \geq 1 - n^{-11r}$.

Assuming that this bound holds,

$$v^{(k_1)}(x^{(2)}, \dots, x^{(r)}) \leq (C_3 r \log n)^{r-2}$$

for all $k_1 \in V_1$ and $x^{(2)}, \dots, x^{(r)}$. By Lemma 6 then

$$\begin{aligned} \sum_{k_1 \in V_1} B^{(k_1)}(x^{(2)}, \dots, x^{(r)})^2 &= \sum_{k_1 \in V_1} \left(u^{(k_1)} \cdot v^{(k_1)}(x^{(3)}, \dots, x^{(r)}) \right)^2 \\ &> n (C_3 r \log n)^{r-1} \end{aligned}$$

with probability at most

$$\exp\left(-\frac{C_3 r n \log n}{18}\right) (4\sqrt{e\pi})^n$$

which is at most n^{-11r} for a suitably large C_3 .

Altogether the bound of the lemma holds with probability $1 - 2n^{-11r} \geq 1 - n^{-10r}$. □

Proof of Claim 13. Consider a fixed choice of the following: 1) k_1, \dots, k_ℓ and 2) $x^{(\ell+1)} \in U_{n_{\ell+1}}, \dots, x^{(r)} \in U_{n_r}$. From Eqn. 8, we have from definition that for every $k_{\ell+1} \in V_{\ell+1}$

$$\|v^{(k_1 \dots k_{\ell+1})}(x^{(\ell+2)}, \dots, x^{(r)})\|_2^2 \leq f_{\ell+1}^2.$$

Therefore, by Lemma 6

$$\begin{aligned} \sum_{k_{\ell+1} \in \text{supp}(x^{(\ell+1)})} B^{(k_1, \dots, k_{\ell+1})}(x^{(\ell+2)}, \dots, x^{(r)})^2 &= \sum_{k_{\ell+1} \in \text{supp}(x^{(\ell+1)})} \left(u^{(k_{\ell+1})} \cdot v^{(k_1 \dots k_{\ell+1})}(x^{(\ell+2)}, \dots, x^{(r)}) \right)^2 \\ &> C_3 r f_{\ell+1}^2 n_{\ell+1} \log n \end{aligned}$$

with probability at most

$$\exp\left(-\frac{C_3 r n_{\ell+1} \log n}{18}\right) (4\sqrt{e\pi})^{n_{\ell+1}}.$$

Taking a union bound over the choice of k_1, \dots, k_ℓ (at most n^r), and the choice of $x^{(\ell+1)} \in U_{n_{\ell+1}}, \dots, x^{(r)} \in U_{n_r}$ (at most $n^{(r-1)n_{\ell+1}}$), the probability that

$$f_\ell^2 > C_3 r f_{\ell+1}^2 \log n$$

becomes at most

$$\exp\left(-\frac{C_3 r n_{\ell+1} \log n}{18}\right) (4\sqrt{e\pi})^{n_{\ell+1}} n^{r n_{\ell+1}}.$$

For large enough C_3 this is at most n^{-12r} . □

Algorithm 1 An Algorithm for Recovering the Clique

Input:

- 1) Graph G .
- 2) Integer $p = |P|$.
- 3) Unit vector x .

Output: A clique of size of p or FAILURE.

1. Calculate $y^{-\lceil r \log n \rceil}(x), \dots, y^{\lceil r \log n \rceil}(x)$ as defined in the indicator decomposition.
 2. For each such $y^{(j)}(x)$, let $S = \text{supp}(y^{(j)}(x))$ and try the following:
 - (a) Find v , the top eigenvector of the $1, -1$ adjacency matrix $A|_{S \times S}$.
 - (b) Order the vertices (coordinates) such that $v_1 \geq \dots \geq v_{|S|}$. (Assuming dot-prod is $\sqrt{1/2}$ below)
 - (c) For $\ell = 1$ to $|S|$, repeat up to $n^{30} \log n$ times:
 - i. Select $10 \log n$ vertices Q_1 at random from $[\ell]$.
 - ii. Find Q_2 , the set of common neighbors of Q_1 in G .
 - iii. If the set of vertices with degree at least $7p/8$, say P' has cardinality p and forms a clique in G , then return P' .
 - (d) Return FAILURE.
-

4 Finding planted cliques

We now turn to Theorem 2 and to the problem of finding a planted clique in a random graph. A random graph with a planted clique is constructed by taking a random graph and then adding every edge between vertices in some subset P to form the planted clique. We denote this graph as $G_{n,1/2} \cup K_p$. Letting A be the r th order subgraph parity tensor, we show that a vector $x \in B(0, 1)$ that approximates the maximum of $A(\cdot)$ over the unit ball can be used to reveal the clique, using a modification of the algorithm proposed by Frieze and Kannan [3].

This implies an interesting connection between the tensor problem and the planted clique problem. For symmetric second order tensors (i.e. matrices), maximizing $A(\cdot)$ is equivalent to finding the top eigenvector and can be done in polynomial time. For higher order tensors, however, the complexity of maximizing this function is open if elements with repeated indices are zero. For random tensors, the hardness is also open. Given the reduction presented in this section, a hardness result for the planted clique problem would imply a similar hardness result for the tensor problem.

Given an x that approximates the maximum of $A(\cdot)$ over the unit ball, the algorithm for finding the planted clique is given in Alg. 4. The key ideas of using the top eigenvector of subgraph and of randomly choosing a set of vertices to “seed” the clique (steps 2a-2d) come from Frieze-Kannan [3]. The major difference in the algorithms is the use of the indicator decomposition. Frieze and Kannan sort the indices so that $x_1 \geq \dots x_n$ and select one set S of the form $S = [j]$ where $\|A|_{S \times S}\|$ exceeds some threshold. They run steps (2a-2d) only on this set. By contrast Alg. 4 runs these steps on every $S = \text{supp}(y^{(j)}(x))$ where $j = -\lceil r \log n \rceil, \dots, \lceil r \log n \rceil$.

The algorithm succeeds with high probability when a subset S is found such that $|S \cap P| \geq C\sqrt{|S| \log n}$, where C is an appropriate constant.

Lemma 14 (Frieze-Kannan). *There is a constant C_5 such that if $S \subseteq [n]$ satisfies $|S \cap P| \geq C_5\sqrt{|S| \log n}$, then with high probability steps a)-d) of Alg. 4 find a set P' equal to P .*

To find such an subset S from a vector x , Frieze and Kannan require that $\sum_{i \in P} x_i \geq C \log n$. Using the indicator decomposition, as in the Alg 4, however, reduces this to $\sum_{i \in P} x_i \geq C\sqrt{\log n}$. Even more

importantly, using the indicator decomposition means that only one element of the decomposition needs to point in the direction of the clique. The vector x could point in a very different direction and the algorithm would still succeed. We exploit this fact in our proof of Theorem 2. The relevant claim is the following.

Lemma 15. *Let B' be set of vectors $x \in B(0, 1)$ such that*

$$|\text{supp}(y^{(j)}(x)) \cap P| < C_5 \sqrt{|\text{supp}(y^{(j)}(x))| \log n}$$

for every $j \in \{-\lceil r \log n \rceil, \dots, \lceil r \log n \rceil\}$. Then, there is a constant C'_1 such that with high probability

$$\sup_{x \in B'} A(x, \dots, x) \leq C'_1 r^{5r/2} \sqrt{n} \log^{3r/2} n.$$

Proof. By the same argument used in the discretization, we have that for any $x \in B'$

$$\begin{aligned} A(x, \dots, x) &\leq (2\lceil r \log n \rceil)^r \max_{x^{(1)} \in Y^{(1)}(x), \dots, x^{(r)} \in Y^{(r)}(x)} A(x^{(1)}, \dots, x^{(r)}) \\ &\leq (2\lceil r \log n \rceil)^r \max_{x^{(1)}, \dots, x^{(r)} \in U'} A(x^{(1)}, \dots, x^{(r)}), \end{aligned} \quad (9)$$

where

$$U' = \{|S|^{-1/2} \chi^S : S \subseteq [n], |S \cap P| < C_5 \sqrt{|S| \log n}\}.$$

Consider an off-diagonal block $V_1 \times \dots \times V_r$. For each $i \in 1 \dots r$, let $P_i = V_i \cap P$ and let $R_i = V_i \setminus P$. Then, breaking the polynomial $A|_{V(\cdot)}$ up as a sum of 2^r terms, each corresponding to a choice of $S_1 \in \{P_1, R_1\}, \dots, S_r \in \{P_r, R_r\}$ gives

$$\max_{x^{(1)}, \dots, x^{(r)} \in U'} A|_{V(x^{(1)}, \dots, x^{(r)})} \leq 2^r \max_{x^{(1)}, \dots, x^{(r)} \in U'} \sum_{S_1 \in \{P_1, R_1\}, \dots, S_r \in \{P_r, R_r\}} A|_{S_1 \times \dots \times S_r}(x^{(1)}, \dots, x^{(r)}). \quad (10)$$

By symmetry, without loss of generality we may consider the case where $S_i = R_i$ for $i = 1 \dots r - \ell$ and $S_i = P_i$ for $i = r - \ell + 1 \dots r$ for some ℓ . Let $\tilde{V} = R_1 \times \dots \times R_{r-\ell} \times P_{r-\ell+1} \times \dots \times P_r$. Then,

$$\max_{x^{(1)}, \dots, x^{(r)} \in U'} A|_{\tilde{V}}(x^{(1)}, \dots, x^{(r)}) = \sum_{k_1 \in R_1} \dots \sum_{k_{r-\ell} \in R_{r-\ell}} \prod_{i=1 \dots r-\ell} x_{k_i}^{(i)} \prod_{i,j:i,j \leq r-\ell} E_{k_i k_j} B^{(k_1, \dots, k_{r-\ell})},$$

where (as defined Eqn. 3)

$$B^{(k_1, \dots, k_{r-\ell})}(x^{(r-\ell+1)}, \dots, x^{(r)}) = \sum_{k_{r-\ell+1} \in P_{r-\ell+1}} \dots \sum_{k_r \in P_r} \prod_{i=r-\ell+1 \dots r} x_{k_i}^{(i)} \prod_{i,j:i,r-\ell+1 < j} E_{k_i k_j}.$$

By the assumption that every $x^{(i)} \in U'$, this value is at most $(C_5 \log n)^{\ell/2}$. Thus,

$$\max_{x^{(1)}, \dots, x^{(r)} \in U'} A|_{\tilde{V}}(x^{(1)}, \dots, x^{(r)}) \leq \sum_{k_1 \in R_1} \dots \sum_{k_{r-\ell} \in R_{r-\ell}} \prod_{i=1 \dots r-\ell} x_{k_i}^{(i)} \prod_{i,j:i,j \leq r-\ell} E_{k_i k_j} (C_5 \log n)^{\ell/2}.$$

Note that every edge $E_{k_i k_j}$ above is random, so the polynomial may be bounded according to Lemma 11. Altogether,

$$\max_{x^{(1)}, \dots, x^{(r)} \in U'} A|_{\tilde{V}}(x^{(1)}, \dots, x^{(r)}) \leq (\max\{C_5, C_3\} \log n)^{r/2}.$$

Combining Eqn. 9, Eqn. 10, and applying Lemma 5 completes the proof with C'_1 chosen large enough. \square

Proof of Theorem 2. The clique is found by finding a vector x such that $A(x, \dots, x) \geq \alpha^r |P|^{r/2}$ and then running Algorithm 4 on this vector. Algorithm 4 clearly runs in polynomial time, so the theorem holds if the algorithm succeeds with high probability.

By Lemma 14 the algorithm does succeed with high probability when $x \notin B'$, i.e. when some $S \in \{\text{supp}(y - \lceil r \log n \rceil(x), \dots, \text{supp}(y - \lceil r \log n \rceil(x))\}$ satisfies $|S \cap P| \geq C_5 \sqrt{|S| \log n}$.

We claim $x \notin B'$ with high probability. Otherwise, for some $x \in B'$,

$$A(x, \dots, x) \geq \alpha^r p^{r/2} > C_0^r r^{5r/2} \sqrt{n} \log^{3r/2} n,$$

which is a low probability event by Lemma 15 if $C_0 \geq C'_1$. \square

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A Proof of Lemma 14

Here, we give a Frieze and Kannan's proof of Lemma 14 for the reader's convenience. First, we show that the top eigenvector of $A|_{S \times S}$ is close to the indicator vector for $S \cap P$.

Claim 16. *There is a constant C such that for every $S \subseteq [n]$ where $|S \cap P| \geq C\sqrt{|S| \log n}$, the top eigenvector v of the matrix $A|_{S \times S}$ satisfies*

$$\sum_{i \in S \cap P} v_i > \sqrt{|S \cap P|/2}$$

Proof. The adjacency matrix A can be written as the sum of $\chi^P \chi^{P^T}$ and a matrix R representing the randomly chosen edges. Let $u = \chi^{S \cap P} / \sqrt{|S \cap P|}$. Suppose that v is the top eigenvector of $A|_{S \times S}$ and let $c = u \cdot v$. Then

$$\begin{aligned} |S \cap P|^{1/2} &= A(u, u) \\ &\leq A|_{S \times S}(v, v) \\ &= c^2 A|_{S \times S}(u, u) + 2c\sqrt{1-c^2} A|_{S \times S}(u, v - cu) + (1-c^2) A|_{S \times S}(v - cu, v - cu) \\ &\leq c^2 |S \cap P|^{1/2} + 3\|R|_{S \times S}\|. \end{aligned}$$

Hence

$$c^2 \geq 1 - 3 \frac{\|R|_{S \times S}\|}{C\sqrt{|S| \log n}}.$$

By taking a union bound over the subsets S of a fixed size, it follows from well-known results on the norms of symmetric matrices ([4, 9], also Lemma 6) that with high probability

$$\|R|_{S \times S}\| = O(\sqrt{|S| \log n})$$

for every $S \subseteq [n]$. Therefore, the theorem holds for a large enough constant C . \square

Next, we show that the clique is dense in the first $8|S \cap P|$ coordinates (ordered according to the top eigenvector v).

Claim 17. *Suppose $v_1 \geq \dots \geq v_n$ and $\sum_{i \in S \cap P} v_i > \sqrt{|S \cap P|/2}$. Then for $\ell = 8|S \cap P|$*

$$|[\ell] \cap P| \geq \frac{|S \cap P|}{8}.$$

Proof of Claim 17. For any integer ℓ ,

$$\begin{aligned} \sqrt{\ell} &\geq \sum_{i \leq \ell} v_i \\ &\geq \frac{\ell}{|S \cap P|} \sum_{i > \ell, i \in P} v_i \\ &= \frac{\ell}{|S \cap P|} \left(\sum_{i \in P} v_i - \sum_{i \leq \ell, i \in P} v_i \right) \\ &\geq \frac{\ell}{|S \cap P|} \left(\sqrt{|S \cap P|/2} - \sqrt{|[\ell] \cap P|} \right). \end{aligned}$$

Thus,

$$\sqrt{|[\ell] \cap P|} \geq \sqrt{|S \cap P|/2} - \frac{|S \cap P|}{\sqrt{\ell}}.$$

Taking $\ell = 8|S \cap P|$ (optimal), we have

$$\sqrt{|\ell \cap P|} \geq \frac{1}{2\sqrt{2}} \sqrt{|S \cap P|}.$$

□

Given this density, it is possible to pick $10 \log n$ vertices from the clique and use this as a seed to find the rest of the clique. When $\ell = 8|S \cap P|$, in each iteration there is at least a

$$8^{-10 \log n} = n^{-30}$$

chance that $Q_1 \subseteq P$. With high probability, no set of $10 \log n$ vertices in P has more than $2 \log n$ common neighbors outside of P in G . The contrary probability is

$$\binom{|P|}{10 \log n} \binom{n}{2 \log n} 2^{-20 \log^2 n} = o(1).$$

Letting Q_2 be the common neighbors of Q_1 in G , it follows that $Q_2 \supseteq P$ and $|Q_2 \setminus P| \leq 2 \log n$. Now, with high probability no common neighbor has degree more than $3|P|/4$ in P , because

$$n \binom{|P|}{10 \log n} \binom{n}{2 \log n} \exp(-|P|/24) = o(1).$$

for $|P| > 312 \log^2 n$.

Thus, with high probability no vertex outside of P will have degree greater than $7|P|/8$ in the subgraph induced by Q_2 .