Extensions of the 'hitchhiker' game-theoretic network formation model

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Observed properties of social networks and the world wide web

- Small average distance and small diameter (“Six degrees of separation”, “Small world”)
- High clustering coefficient
- Power-law degree distribution (scale-freeness)
- Assortativity: “Rich men’s club”
- Existence of densely connected core (most shortest paths go through 5% most active vertices — hubs)
- Some dynamic features (not in the focus of current work)
- etc.
Types of social and web graph models

- Random graphs (Watts-Strogatz, Barabási-Albert, Bollobás-Riordan, Buckley-Osthus, Lattanzi-Sivakumar, geometric random graphs etc.);
- Game-theoretic network formation models (Jackson-Wolinsky, Bala-Goyal etc.) where agents form connections directly;
- Game-theoretic approach with “hitchhiking” (Borgs, Chayes, Ding, Lucier 2010): agents may purchase connections for other agents as a side effect
Affiliation networks

Network formation through affiliations:

- Initially: bipartite graph between agents and societies;
- Network formation: a connection is formed between two agents participating in a same society.
Affiliation networks

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Example:

```
A
/   \\  \
B   C D
/  |  |
E   F
   G
```

- Workshop
- Hiking
- Tennis
Affiliation networks

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Workshop

Hiking

Tennis

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A -- B -- C -- D

E -- F

G

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```
C -- D -- E

A -- B

G

```
Hitchhiking approach: gatherings

- $V$ — the set of agents
- Each agent may organize an arbitrary number of gatherings $P \subset V$ with rates $r > 0$.
- Gatherings are costly: the fixed cost for gathering of rate $r$ is $rb$, the marginal cost for one extra participant is $rc$.
- So, $C(P_v,i) = r_{v,i}(c|P_v,i \setminus \{v\}| + b)$, where $v$ is an agent and $i$ is the identifier of gathering. Note that self-inviting is costless.
A link between $u$ and $v$ is formed if they both participate in gatherings with total rate greater than 1.

No matter who have organized these gatherings: $u$, $v$ or a third agent.

Each link delivers utility $a$ to both agents.

Formally: $(vu) \in E$ if $\sum_{(w,i): u,v \in P_{w,i}} r_{w,i} \geq 1$

$N_v = \#\{u: (vu) \in E\}$.

$U_v = aN_v - \sum_i C(P_v,i)$. 
Hitchhiking approach: equilibria

- The situation is a one-shot game
- The solution concept: Nash equilibrium
- The main question: what networks are supportable in equilibrium for what parameters?
- Theorem: star configuration is not supportable.
- Theorem: any supportable graph has sufficiently high clustering coefficient.
- Theorem: power-law is supportable.
- But...
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- Theorem: star configuration is not supportable.
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- But... there are too many equilibria.
Crucial multiplicity of equilibria

**Theorem**
If at least one non-empty configuration is supportable then the complete graph $K_n$ is supportable.

**Proof idea**
The least average edge cost is reached when any gathering coincides with $N$. It can be realized for all agents simultaneously if every agent organizes the grand gathering with rate $\frac{1}{n}$. In any equilibrium some agent has greater edge cost and since it is profitable for him then so does the symmetric case with grand gatherings.
How to reduce the number of equilibria: some ideas

- Upper bound on the size of gatherings
- Convex/superadditive/supermodular costs of gathering
- Harder to make a connection in bigger gatherings
- Heterogeneous costs of organizing gathering
- Costly participation in a gathering
- Heterogeneous profits
- Profits from not only immediate neighbors
- Dynamic network formation
- Your suggestions?
Extension: heterogeneous costs

- Suppose that for any two agents $u$ and $v$ there is a “distance” $\rho_{uv} \geq 0$, such that $\rho_{uu} = 0$.
- Distance is not necessarily a metric or even a symmetric measure.
- $\rho_{uv}$ is treated as the cost of inviting $v$ for the agent $u$.
- $C_v(P) = r(c \sum_{v \in P} \rho_{uv} + b)$.
- The initial model is a subcase for $\rho_{uv} = 1 - \delta_{uv}$.
Heterogeneous costs: geometrical interpretation

- Inspired by economic geography models
- Every agent $u$ lives in a point $x_u$ in some metric space representing geographical location or some taste vector
- $\rho_{uv}$ equals to the distance between $x_u$ and $x_v$
- The simplest case: uniform distribution on a circle
- A must-be equilibrium: every agent $u$ invites a segment $[x_u - d, x_u + d]$ with some rate $r$
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- A must-be equilibrium: every agent $u$ invites a segment $[x_u - d, x_u + d]$ with some rate $r$
- Actual symmetric equilibrium: every agent $u$ invites a segment $[x_u + d_1, x_u + d_2]$ with rate 1!
- Do we need more dimensions?
Heterogeneous costs among those who invite

- In reality, some people organize many different gatherings, others organize nothing
- It might be that the cost is varying across inviter: \( \rho_{uv} = \rho_u \)
- In turns out that conclusions should be pretty similar to the initial model
- Interesting result: for maximal costs that admit a non-trivial equilibrium all agents bear the same costs in this equilibrium
- This extension may be good for comparative statics
Extension: heterogeneous profits

- Suppose that for any two agents \( u \) and \( v \) there is a “potential” \( \pi_{uv} \geq 0 \)
- Potential is not necessarily symmetric.
- \( \pi_{uv} \) is treated as the profit of having a connection to \( v \) for the agent \( u \).
- \( U_v = a \sum_{(uv) \in E} \pi_{uv} - \sum C_v(P) \).
- The initial formalisation is a subcase for \( \pi_{uv} = 1 - \delta_{uv} \).
Plans

- More detailed investigation of the case $\rho_{uv} = \rho_u$
- Analysis of the case $\rho_{uv} = \rho_v$ (some agents are “slow to move”, others are more agile)
- Analysis of heterogeneous profits and combination of heterogeneous costs and profits
- Some numerical simulations
- Other extensions of the model
Conclusions

- Modeling a social graph is a great challenge
- Social networks usually are affiliation network, so it is natural to model such types of networks
- Game-theoretic models seem to be more natural but are currently less explanatory
- It is hard to combine tractability and accordance with the reality
Thank you for your attention!
Questions? Suggestions?
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