

Some Computational and Game-Theoretic Issues in Nash and Nonsymmetric Bargaining Games

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Abstract

Motivated by recent insights on the two solution concepts of Nash and nonsymmetric bargaining games, obtained from an algorithmic study of these notions [Vaz09], we take a fresh look at understanding their quality and robustness by subjecting them to several “stress tests”.

Our tests are quite basic, e.g., we ask whether the solutions delivered are efficient, fair, respond in a desirable manner when agents change their disagreement points or play with a subset of the agents, and whether they lend themselves to efficient computability.

Our main conclusion is that imposing *submodularity*, a natural economies of scale condition, on Nash and nonsymmetric bargaining games endows them with several desirable properties.

1 Introduction

Bargaining was first modeled as a game in John Nash’s seminal 1950 paper [Nas50], using the framework of game theory given a few years earlier by von Neumann and Morgenstern [vNM44]. Bargaining is perhaps the oldest situation of conflict of interest, and since game theory develops solution concepts for negotiating in such situations, it is not surprising that this paper led to a theory (of bargaining) that lies today at the heart of game theory (e.g., see [Kal85, TL89, OR90, aum94]).

Nash’s solution to the bargaining game is the unique point in the feasible set that satisfies 4 reasonable axioms. A well known generalization of Nash bargaining was obtained by Kalai [Kal77] by removing Nash’s axiom of symmetry. Unlike the Nash bargaining solution, which is unique, a game has infinitely many nonsymmetric bargaining solutions. One can single out one of these by specifying the *clout* of each player; for the purposes of computation, we need to assume that the clouts of players are all rational, or equivalently integral, numbers. Kalai’s main theorem implies that the nonsymmetric solution to an n -person bargaining game, with integral clouts, corresponds to a Nash bargaining solution to a game with a larger number of players which is obtained by replicating each of the n players as many times as her clout.

Over the years, the theory of bargaining developed along two major aspects. The more prominent one was the axiomatic approach whose goal was to characterize the solution concept that results from assuming a given set of axioms. The second was to determine the quality of the basic solution concepts of this theory by subjecting them to what may be viewed as “stress tests”.

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These results determine whether or not a particular solution concept satisfies a particular test on *all instances* of the bargaining game.

We list the properties of Nash bargaining games that have been studied so far (e.g., see [Thob]). Thomson [Tho87] shows that all such games satisfy *monotonicity*, i.e., if the disagreement utility of a player is increased, then her utility cannot decrease. He also studied *strong monotonicity*, i.e., if the disagreement utility of one player is increased, then any other player’s utility should not increase. He showed that Nash bargaining games do not always satisfy strong monotonicity. Thomson [Tho83] also defines the notion of *population monotonicity*, i.e., when bargaining with a proper subset of the players, an agent’s utility should be non-decreasing, and shows that the Nash Bargaining solution doesn’t satisfy this property.

Although useful, such an “all-or-nothing” results can give only limited information on the quality of a solution concept. In particular, the instances of interest may be highly structured and therefore may pass the test even though the entire solution concept does not. Motivated by recent insights on the two solution concepts of Nash and nonsymmetric bargaining games, obtained from an algorithmic study of these notions [Vaz09], we take a fresh look at understanding their quality and robustness. We deviate from the earlier line of work in two respects. First, we use the classification of these games given in [Vaz09], to determine if specific classes of these games satisfy a given property. Second, we define and study new tests that follow naturally from an algorithmic game theory perspective.

Our main conclusion is that imposing *submodularity*, a natural economies of scale condition, on Nash and nonsymmetric bargaining games endows them with several desirable properties. Our new tests are of a basic nature. For instance, we ask whether the solutions yielded by these two concepts are “efficient” and “fair”, after giving reasonable, quantitative definitions of efficiency and fairness. We also define and study the notion of *surplus monotonicity*, i.e., if the disagreement utility of a player is increased, then her surplus utility, over and above the disagreement utility, increases.

As stated above, an important catalyst for our results is the insights gained from a deliberately *combinatorial* algorithmic approach to the Nash and nonsymmetric bargaining games insisted on in [Vaz09]. Besides the algorithms themselves, these include the combinatorial structural insights gained in obtaining the algorithms, and the classification of these games obtained by imposing restrictions on the the set of feasible utilities (or equivalently, imposing conditions on the constraints in the convex programs that capture the solutions to these games).

Bargaining is a basic mechanism for trading that has been used by mankind since times immemorable. Indeed, one can even see portals on the Internet emerging that use this mechanism, e.g., iOffer.com is a website similar to eBay except that it uses bargaining instead of auctions to sell goods, and priceline.com now allows customers to name their price for airline tickets and hotel reservations. The Internet is particularly well suited for enabling individual negotiations to take place and one can expect a more wide-spread use of bargaining on it. Clearly, these new applications, involving both game theoretic and algorithmic issues, are best studied in the “traditional” style of algorithmic game theory.

1.1 Detailed description of our results

In order to describe our results more formally, we need to recall some classes of Nash and non-symmetric bargaining games defined recently in [Vaz09]. LNB (Linear Nash and Nonsymmetric Bargaining Games) is the class of games whose feasible set of utilities is defined by finitely many packing constraints. UNB (Uniform Utility Nash and Nonsymmetric Bargaining Games) is the subclass of LNB in which for each available resource, each agent who uses this resource uses it

in the same way, i.e., the coefficients in the packing constraints are 0/1. Clearly, only $2^{|A|}$ such constraints are needed, where A is the set of agents – one for each subset of A . We can now view the right hand sides of these constraints as being given by a set valued function over the power set of A . If this function is submodular, the game is said to be in the subclass SNB (Submodular Utility Nash and Nonsymmetric Bargaining Games) of UNB.

Several of our results show that a game in UNB has a certain property iff it is in SNB. Equivalently, these results characterize SNB within UNB. Submodularity is a natural economies of scale condition and the class SNB contains important natural games. These results should not come as a surprise – submodularity has been exploited in similar ways in the past in game theory, e.g., see Moulin and Shenker [MS97], showing that a cross-monotonic cost sharing method gives rise to a budget balanced and group strategyproof cost sharing mechanism iff the cost function is submodular, and the results of [JV07], giving analogous characterizations of Eisenberg-Gale markets defined via submodular functions.

1.2 Computational results

In this paper, we first continue a computational study of Nash and nonsymmetric bargaining games. The solution to any such game is the optimal solution to a convex program that involves maximizing a concave function on a convex set (the set of feasible utility vectors). Therefore, if we can obtain a separation oracle for this convex program, we can approximate the Nash Bargaining solution in polynomial time to any required degree of accuracy using the ellipsoid algorithm. This leads to the question: can a separation oracle be implemented in polynomial time? We show that the answer is “yes” for any game in SNB and we show that there are games in UNB for which no randomized algorithm can implement separation oracle in poly time with any reasonable success probability; the first result compliments the result in [Vaz09] giving a combinatorial strongly polynomial time algorithm for finding the solution to any game in SNB.

In game theory, it is customary to assume that the given instance of Bargaining game is *feasible*, i.e., the disagreement point lies inside the feasible utility polytope. In our parlance, game theorists only look at the promise problem, where the promise is that the game is feasible. From an algorithmic viewpoint, the problem of deciding feasibility reduces to solving the promise problem. Hence, we might as well solve the decision and search problems explicitly.

1.3 Game-theoretic properties studied

We define *efficiency* or the *price of bargaining* of a game in a way that is analogous to the notion of price of anarchy [KP99]: it is the ratio of the net worth created by the Nash bargaining solution and that achieved by the most efficient solution that can be obtained in a centralized manner. Recall that the price of anarchy does the same for the solution yielded by a Nash equilibrium. We say that a Nash bargaining game is *efficient* if its price of bargaining is 1.

We will say that a Nash bargaining game is *strongly monotone* if whenever one player increases his disagreement utility, no other player’s utility can increase in the resulting Nash or nonsymmetric bargaining solution.

We compare the max-min and min-max fairness of the Nash or nonsymmetric bargaining solution in relation to all feasible solutions that are Pareto optimal. The solution will be said to be *fully fair* if it satisfies both max-min and min-max fairness. This is the most stringent notion of fairness studied in literature.

We define the *surplus utility* of an agent to be the utility she gets in a Nash or nonsymmetric bargaining solution, over and above her disagreement utility. We say that a Nash or nonsymmetric bargaining game is *surplus monotone* if increasing the disagreement utility of any agent does not lead to an increase in her surplus utility.

Consider the utilities derived by an agent in the bargaining solution when playing with various subsets of the entire set of agents. We say that a Nash bargaining game is *population monotone* if the utility of any agent in the bargaining solution can only increase when playing with a subset of the agents. Similarly, we say that a nonsymmetric bargaining game is *population monotone* if the utility of any agent in the bargaining solution can only increase if the utilities of other agents are decreased; this notion is also independently defined by Thomson [Thoa].

For each of the properties stated above, except surplus monotonicity, we show that a game in UNB has this property iff it is also in SNB, i.e., these properties characterize SNB within UNB.

2 Nash and Nonsymmetric Bargaining Games

For a set of agents A , a Nash bargaining game is defined by a pair $(\mathbf{c}, \mathcal{P})$, where $\mathcal{P} \subseteq \mathbb{R}_+^{|A|}$ is a compact and convex set which defines the feasible set of utilities of all the agents, and $\mathbf{c} \in \mathcal{P}$ is known as the disagreement point which defines the amount of utility each agent will get if the bargaining process fails.

Nash [Nas50] defined the bargaining solution $\mathbf{u} \in \mathcal{P}$ of this game to be the one which satisfies four axioms:

1. **Pareto optimality:** No point in \mathcal{N} can weakly dominate $\mathbf{u}^*(\mathbf{c})$.
2. **Invariance under transformation of utilities:** The affine transformation of the utilities leads to the same affine transformation on $\mathbf{u}^*(\mathbf{c})$.
3. **Symmetry:** If the agents are renumbered, then it suffices to renumber the coordinates of $\mathbf{u}^*(\mathbf{c})$.
4. **Independence of irrelevant alternatives:** If $\mathbf{u}^*(\mathbf{c})$ is the solution for $(\mathcal{N}, \mathbf{c})$ where $\mathbf{c}, \mathbf{u}^*(\mathbf{c}) \in \mathcal{S}$ and $\mathcal{S} \subseteq \mathcal{N}$, then $\mathbf{u}^*(\mathbf{c})$ is also the solution for $(\mathcal{S}, \mathbf{c})$.

Nash proved that there is a unique point in \mathcal{P} which satisfies these axioms, and moreover this point ($\mathbf{u} \in \mathcal{P}$) is the one that maximizes $\prod_{i \in A} (u_i - c_i)$ or equivalently $\sum_{i \in A} \log(u_i - c_i)$.

In the nonsymmetric bargaining game, each agent i has a positive bargaining weight or clout α_i which represents his relative bargaining power. The bargaining solution in this case is the point in the feasible region \mathcal{P} which maximizes the objective function $\sum_{i \in A} \alpha_i \log(u_i - c_i)$.

2.1 Uniform Utility Nash and Nonsymmetric Bargaining Games

The class Linear Nash and Nonsymmetric Bargaining Games (LNB), defined in [Vaz09], consists of games whose feasible set \mathcal{P} is defined by a finite number of linear packing constraints. The main focus of our paper will be on a natural subclass of LNB called Uniform Utility Nash and Nonsymmetric bargaining games (or UNB) which was also defined in [Vaz09]. In these games, the coefficients of the variables in the linear packing constraints are either 0 or 1. Clearly there can be

at most $2^{|A|}$ such constraints, thus a function of the form $v : 2^A \rightarrow R^+$ uniquely encodes a feasible set in UNB games. A UNB game is called an SNB if the function v is a submodular function. In Section 3, we give examples of UNB and SNB games.

Now given a disagreement point \mathbf{c} , and a fixed set of agents $T \subseteq A$, the solution to a uniform utility Nash bargaining game is captured by the following convex program:

$$\begin{aligned} \max \quad & \sum_{i \in T} \log(u_i - c_i) \\ \text{s.t.} \quad & \forall S \subset T : \sum_{i \in S} u_i \leq v(S) \\ & \forall i \in T : u_i \geq 0 \end{aligned} \tag{1}$$

For uniform utility nonsymmetric bargaining games, the objective functions that needs to be maximized is $\sum_{i \in T} \alpha_i \log(u_i - c_i)$.

For a fixed function $v : 2^A \rightarrow R^+$, we will define a family of games $F(v)$ to be the set of all Nash bargaining games for various choices of disagreement points \mathbf{c} and set $T \subseteq A$. An instance $(\mathbf{c}, T) \in F(v)$ will refer to a particular Nash bargaining game in $F(v)$ with a fixed set T and disagreement point \mathbf{c} . We will use $u_i^*(\mathbf{c}, T)$ to denote the utility of the player i in the Nash's solution for the bargaining game $(\mathbf{c}, T) \in F(v)$.

Similarly, for a fixed function $v : 2^A \rightarrow R^+$ and a positive bargaining weight vector $\alpha = (\alpha_i)_{i \in A}$, we can define $F(v, \alpha)$ to be the set of all Nonsymmetric bargaining games for various choices of disagreement points \mathbf{c} and set $T \subseteq A$.

We will assume that the following two natural conditions are satisfied by the function v :

1. **Non degenerate:** $v(\emptyset) = 0$.
2. **Non redundancy of sets:** $\forall S \subseteq A$, there exists a feasible utility vector \mathbf{u} such that set S is *tight* w.r.t. \mathbf{u} , i.e. $\sum_{i \in S} u_i = v(S)$. It is easy to see (using duality) that this property is equivalent to the property that v satisfies fractional covering property i.e. $\forall S, v(S) \leq \sum_{B \subseteq A} v(B) x_B$, where x_B 's are such that $\forall i \in S: \sum_{B: i \in B} x_B \geq 1$.

We will call such functions to be *valid* functions. Note that the second condition above implies 1) *Monotonicity*: for any $Z_1 \subset Z_2 \subseteq A$, we have $v(Z_1) \leq v(Z_2)$, and 2) *Complement freeness*: $v(Z_1 \cup Z_2) \leq v(Z_1) + v(Z_2)$.

In this paper, we are interested in the following properties of UNB games.

1. **Price of Bargaining:** For any valid function $v : 2^A \rightarrow R^+$, we define the *Price of bargaining* of $F(v)$ to be $\min_{(\mathbf{c}, T) \in F(v)} \frac{u^*(\mathbf{c}, T)}{v(T)}$, where $u^*(\mathbf{c}, T) = \sum_i u_i^*(\mathbf{c}, T)$. In particular, we are interested in finding out if UNB games have *Price of bargaining* equal to 1 or not. Thus, this property captures the inefficiency of the bargaining solution.
2. **Fairness:** For any instance $I = (\mathbf{c}, T) \in F(v)$, define *core*(I) to be the set of all feasible Pareto optimal solutions. For any vector \mathbf{u} , let \mathbf{u}_{dec} be the vector obtained by sorting the components of \mathbf{u} in decreasing order. A vector \mathbf{x} *min-max dominates* \mathbf{y} if \mathbf{x}_{dec} is lexicographically smaller

than \mathbf{y}_{dec} . Also let \mathbf{u}^* be bargaining solution of instance I . Instance I is said to be *min-max fair* if the vector $\mathbf{u}^* - \mathbf{c}$ *min-max dominates* $\mathbf{y} - \mathbf{c}$ for all $\mathbf{y} \in \text{core}(I)$. $F(v)$ is said to be *min-max fair* if all the instances in $F(v)$ are min-max fair. Similarly, we define the notion of *max-min fairness*.

3. **Strong monotonicity:** For any valid function $v : 2^A \rightarrow R^+$, we say that $F(v)$ is *strongly monotone* if, for all games in $F(v)$, the following property holds: On increasing the disagreement utility c_i of an agent i , the bargaining solution doesn't increase the utility for any other agent j , where $j \neq i$.
4. **Surplus monotonicity:** For any valid function $v : 2^A \rightarrow R^+$, we say that $F(v)$ is *surplus monotone* if the surplus (i.e. $u_i^* - c_i$) of any agent i in the bargaining solution does not increase if her disagreement utility c_i increases. Formally, for any instance (\mathbf{c}, T) and (\mathbf{c}', T) , $T \subseteq A$, where $\mathbf{c}' = \mathbf{c}$ except that $c'_i > c_i$, then we have

$$u_i^*(\mathbf{c}', T) - c'_i \leq u_i^*(\mathbf{c}, T) - c_i.$$

5. **Population monotonicity:** For any valid function $v : 2^A \rightarrow R^+$, we say that $F(v)$ is *population monotone* if, for any $T_1 \subset T_2 \subseteq A$ and any agent $i \in T_1$, agent i cannot obtain more utility in the bargaining solution of instance (\mathbf{c}, T_2) than in the bargaining solution of (\mathbf{c}, T_1) , i.e. $u_i^*(\mathbf{c}, T_1) \geq u_i^*(\mathbf{c}, T_2)$.

The main result of this paper is that for each of the above properties, other than surplus monotonicity, the property holds for $F(v)$ if and only if v is submodular.

All of the above definitions can be naturally extended for $F(v, \alpha)$ - the family of nonsymmetric bargaining games. Most of our results also extend naturally to this family of games; we point out the key differences in section 10. For the case of *population monotonicity*, one can instead consider a stronger version, which says that bargaining utility of an agent shouldn't decrease as the clout of some other agent is decreased. We again show that $F(v, \alpha)$ satisfy population monotonicity w.r.t clouts if and only if v is submodular.

Thus each of the above properties, other than surplus monotonicity, characterize SNB within UNB.

3 Examples of UNB and SNB games

We give few natural examples of UNB and SNB games here. Examples 1 and 4 illustrates games which are in SNB; example 2 illustrates a game which is in UNB but not in SNB; example 3 illustrates a game which is in LNB but not in UNB. In each of these examples we only give the feasible set of utilities. The disagreement point \mathbf{c} could be any point inside the feasible set.

Example 1 Consider the set A of n agents trying to share 1 dollar amongst themselves. Suppose the utility of agent i is precisely the amount of money he gets in the final solution. Then the feasible set of utilities is precisely

$$\mathcal{P} := \{\forall S \subseteq A : \sum_{i \in S} u_i \leq 1; \quad u_i \geq 0, \quad \forall i\}$$

Example 2 Consider a set of three agents $\{1, 2, 3\}$ who are trying to share three divisible items $\{A, B, C\}$ of 1 unit each. Agent 1 is interested in *equal amounts* of item A and B but doesn't

care about item C . Given amount x of both A and B , he values the bundle at $30x$; however the maximum utility he derives is 20 units. Formally, given a, b, c amounts of goods A , B and C respectively to agent i , his utility is given by

$$u_1(a, b, c) = \min\{20, 30 \cdot \min(a, b)\}$$

Similarly, we let $u_2(a, b, c) = \min\{20, 30 \cdot \min(b, c)\}$ and $u_3(a, b, c) = \min\{20, 30 \cdot \min(c, a)\}$. It is not difficult to see that the set of feasible utilities in this case is

$$\mathcal{P} := \{u_i \leq 20, \forall i; u_i + u_j \leq 30, \forall i, j, \text{ and } i \neq j; u_1 + u_2 + u_3 \leq 45; u_1, u_2, u_3 \geq 0\}$$

Example 3 (*Sharing arcs of a network*) Consider the following directed network $N = (V, A)$ where the arcs have capacities and two agents control two source-sink pairs (s_1, t_1) and (s_2, t_2) respectively. There are two arc-disjoint directed paths from s_1 to t_1 each of length 3 – (s_1, a, b, t_1) and (s_1, x, y, t_1) , where a, b, x, y are nodes in the network; and there is a single directed path from s_2 to t_2 – (s_2, a, b, x, y, t_2) . The only arcs that have capacities are (a, b) and (x, y) and each has a capacity of 1.

The utility of agent i is the amount of flow that goes from source s_i to sink t_i respecting the capacity constraints. Observe that when agent 2 sends x units of flow from s_2 to t_2 , he “eats up” x units on *each* path of agent 1. Moreover, if agent 2 were absent, agent 1 could’ve sent a flow of 2 units (on each of its path) while agent 2 could send a maximum of 1 unit of flow only in the absence of agent 1. That is, the two agents do not share the resources in an uniform manner. The feasible polytope of utilities in this example is given by

$$\mathcal{P} := \{(u_1, u_2) : 0 \leq u_1 \leq 2, 0 \leq u_2 \leq 2; u_1 + 2u_2 \leq 2\}$$

Example 4 (*Sharing arcs of a network with single-source*) Consider a directed network $N = (V, A)$ with capacities on arcs. Let there be a source s in the network. Suppose there is a set of n agents A . Each agent controls a sink t_i and is interested in receiving flow from the source s . The utility of an agent i is the amount of flow that goes from source s to sink t_i . For any set S of agents, let $v(S)$ be the size of min cut separating source s and set S . Note that the function v is a submodular function [Meg74]. Now the feasible polytope of utilities is given by

$$\mathcal{P} := \{\forall S \subseteq A \sum_{i \in S} u_i \leq v(S); \forall i u_i \geq 0\}$$

4 Preliminaries

For any valid function v , we say that S is *tight* w.r.t. \mathbf{u} if $\sum_{i \in S} u_i = v(S)$. Let \mathbf{u}^* be the solution to the convex program given in (1). Then by KKT conditions, there must exist variables $\{p_S, \forall S \subseteq T\}$ such that:

1. $\forall S \subseteq T, p_S \geq 0$.
2. $\forall S \subseteq T, p_S > 0 \Rightarrow \mathbf{u}^*$ makes set S tight.
3. $\forall k \in T$, we have $u_k > c_k \Rightarrow \sum_{S:k \in S} p_S = \frac{1}{u_k^* - c_k}$. (or $\sum_{S:k \in S} p_S = \frac{\alpha_k}{u_k^* - c_k}$ for the case of nonsymmetric bargaining game)

We will call p_S to be the price of set S .

Now we give some properties of the submodular and non-submodular functions which will be used in our proofs.

Property 4.1. *Given a valid submodular function $v : 2^A \rightarrow R_+$, and a utility vector \mathbf{u} , if $Z_1, Z_2 \subseteq A$ are tight sets w.r.t. \mathbf{u} , then $Z_1 \cup Z_2$ and $Z_1 \cap Z_2$ are also tight sets w.r.t. \mathbf{u} .*

Proof. $u(Z_1 \cup Z_2) + u(Z_1 \cap Z_2) = u(Z_1) + u(Z_2) = v(Z_1) + v(Z_2) \geq v(Z_1 \cup Z_2) + v(Z_1 \cap Z_2) \geq u(Z_1 \cup Z_2) + u(Z_1 \cap Z_2)$, where the second equality follows from the tightness of Z_1 and Z_2 . \square

By using the uncrossing argument and the above property, we get the following corollary.

Corollary 4.1. *Given any valid submodular function v , and $(c, T) \in F(v)$ (or $(c, T) \in F(v, \alpha)$ for the nonsymmetric case), we can choose the prices for all subsets of T in the KKT conditions, such that the tight sets with positive prices form a nested set family, i.e. $T = T_1 \supset T_2 \supset \dots \supset T_k \supset T_{k+1} = \emptyset$.*

We next state a property of non-submodular valid functions which enhances the following theorem of [CD07].

Theorem 4.1. [CD07] *Given any valid non-submodular function v , there exists a set $S \subset A$, $i, j \in A \setminus S$, $l \in S$ and a feasible utility vector \mathbf{u} such that:*

1. $S, S \cup \{i\}, S \cup \{j\}$ are all tight w.r.t \mathbf{u} .
2. No set containing both i and j is tight.
3. All tight sets containing either i or j , must contain the agent l with $u_l > 0$.

The following is the property we will use in most of our proofs.

Property 4.2. *Given any valid non-submodular function v , there exists a set $S \subset A$, $i, j \in A \setminus S$, $l \in S$ and a feasible utility vector \mathbf{u}' such that:*

- (a) v is submodular on $S \cup \{i\}$.
- (b) $S \cup \{i\}, S \cup \{j\}$ are both tight.
- (c) Let $T = S \cup \{i, j\}$, \mathcal{F}_k denotes the set of all subsets which contains k and is tight w.r.t \mathbf{u}' . We have

$$\mathcal{F}_l = \mathcal{F}_i \cup \mathcal{F}_j, \mathcal{F}_i \cap \mathcal{F}_j = \emptyset \quad .$$

- (d) $u_k > 0, \forall k \in T$.

Proof. Let S, i, j, l , and \mathbf{u} be as in the theorem 4.1. We will modify the utility vector \mathbf{u} to another vector \mathbf{u}' such that \mathbf{u}' satisfies our requirements.

First of all, if there is any $r \in S$ such that $u_r = 0$, we remove the agent r from set S : let $S' = S \setminus \{r\}$. We will show that S', i, j, l, \mathbf{u} still satisfies the conditions in Theorem 4.1.

For condition 1, $S' \cup \{i\}$ is tight because

$$\sum_{k \in S' \cup \{i\}} u_k = \sum_{k \in S \cup \{i\}} u_k = v(S \cup \{i\}) \geq v(S' \cup \{i\}) \geq \sum_{k \in S' \cup \{i\}} u_k.$$

The second and third equalities are due to the tightness of $S \cup \{i\}$ and the monotonicity of v respectively. Similarly, $S' \cup \{j\}$ is also tight.

For condition 2, Any tight set $Z \subseteq S' \cup \{i, j\}$, $i, j \in Z$, is also a subset of $S \cup \{i, j\}$ containing both i, j , which cannot be tight.

For condition 3, any tight set in $S' \cup \{i, j\}$ containing either i or j is still tight set in $S \cup \{i, j\}$, so must contain l , $u_l > 0$.

By the above argument, we can remove all the agents in S with zero utility in \mathbf{u} without violating the conditions in Theorem 4.1. Without loss of generality, we may assume that for $\forall k \in S$, $u_k > 0$. Let $T = S \cup \{i, j\}$, we consider vector \mathbf{u} as its restriction to T .

In the proof of Theorem 4.1, the author took $S \cup \{i\}$ as the smallest set violating the submodularity, hence v is submodular on $S \cup \{i\}$. Thus condition (a) holds.

Let $\mathbf{u}' = \mathbf{u}$ except that

$$u'_i = u_i + \epsilon, \quad u'_j = u_j + \epsilon, \quad \text{and} \quad u'_l = u_l - \epsilon.$$

In order to make this vector \mathbf{u}' feasible, we need to set ϵ small enough such that

$$\epsilon < \min\{\epsilon_0, u_l/2\}, \quad \text{where} \quad \epsilon_0 := \min_{\text{nontight } Z \subseteq T} \frac{(v(Z) - \sum_{k \in Z} u_k)}{2}.$$

For any nontight set $Z \subseteq T$ w.r.t \mathbf{u} ,

$$\sum_{k \in Z} u'_k < \sum_{k \in Z} u_k + 2\epsilon_0 \leq v(Z),$$

therefore Z is still nontight w.r.t \mathbf{u}' . For any tight set $Z \subseteq T$ containing either i or j , it must also contain l , hence $\sum_{k \in Z} u'_k = \sum_{k \in Z} u_k$, hence Z is still tight. In particular, (b) holds. If it does not contain i or j , then $\sum_{k \in Z} u'_k \leq \sum_{k \in Z} u_k = v(Z)$. Thus \mathbf{u}' is feasible. Further more, we have $u'_l > 0$ since $\epsilon < \frac{u_l}{2}$. Therefore, $u'_k > 0, \forall k \in T$, which is (d).

For condition (c), obviously, $\mathcal{F}_i \cap \mathcal{F}_j = \emptyset$ and $\mathcal{F}_i \cup \mathcal{F}_j \subseteq \mathcal{F}_l$. On the other hand, for any $Z \subset T$, $i, j \notin Z$, $l \in Z$, we have $\sum_{k \in Z} u'_k = \sum_{k \in Z} u_k - \epsilon < v(Z)$, hence Z is not tight w.r.t \mathbf{u}' . Thus if $Z \in \mathcal{F}_l$, then either $i \in Z$ or $j \in Z$. Hence $\mathcal{F}_l = \mathcal{F}_i \cup \mathcal{F}_j$. \square

5 Computational issues in UNB

As mentioned in Section 2, the solution of Nash and nonsymmetric bargaining games is an optimal solution to a certain convex program. This raises the question of obtaining the bargaining solution by solving the convex program. Using the ellipsoid method, if we can obtain a polynomial time *separation oracle*, then we can approximate the solution to within ϵ in time bounded by a polynomial in the number of agents and $\log(1/\epsilon)$. Given a point in \mathbf{R}_+^n , for the case of an n -player game, such an oracle that given utilities of agents decides if it is a feasible utility vector and if not, provides a separating hyperplane.

In this section, we first consider the following *feasibility problem for UNB*: Given a valid function v , an instance $(\mathbf{c}, T) \in F(v)$ and a utility vector \mathbf{u} , decide whether \mathbf{u} is a feasible utility vector or not. We use $\langle v, (\mathbf{c}, T), \mathbf{u} \rangle$ to denote an instance of the feasibility problem.

Since function v is defined over an exponentially large set, we must assume that it is given via an oracle, i.e. given any $S \subseteq T$, the oracle returns $v(S)$ in unit time. Obviously, any polynomial time algorithm for the feasibility problem must make at most a polynomial number queries to the value oracle. The following theorem shows this is always possible for SNB games although not for all UNB games.

Theorem 5.1. *For the feasibility problem, we have the following:*

1. *Any (randomized) algorithm to solve the Feasibility Problem in UNB with success probability larger than $1/\text{poly}(|T|)$ must make exponentially many queries.*
2. *The Feasibility Problem in SNB can be solved by a polynomial time algorithm.*

Proof. (1) We will give a distribution over different instances, and show that no deterministic algorithm can successfully solve the feasibility problem with probability $1/\text{poly}(|T|)$ in polynomial number of queries. By Yao's min-max lemma, this would imply that any randomized polynomial time algorithm can solve the *Feasibility Problem in UNB* with exponentially small success probability only.

Let function v be defined as $v(S) = |S| + C$, $\forall S \subseteq T$, and some large constant C . Also let $\mathbf{c} = \mathbf{0}$, and $\mathbf{u} = (C + 1, 1, 1, \dots, 1)$. For each $S \subseteq T$, let v^S be the function which equals v except that $v^S(S) = |S| + C - \epsilon$ for some constant ϵ . Constant C is chosen large enough so that all the functions $v^S(\cdot)$ are valid functions. Now of all sets S containing the first element, pick one at random and consider the corresponding instance $\langle v^S, (\mathbf{c}, T), \mathbf{u} \rangle$. For any deterministic algorithm to succeed on this instance it must find out the set S for which $v^S(S) = |S| + C - \epsilon$ otherwise the function $v^S(\cdot)$ could very well have been the function $v(\cdot)$ for which \mathbf{u} is feasible. Now any deterministic algorithm which makes polynomially many queries can only check polynomially many sets S . But since we pick the set S containing the first element at random, the probability of success of this deterministic algorithm is exponentially small.

(2) First, for each $i \in T$, we check whether $u_i > c_i$ or not. If there exists i such that $u_i \leq c_i$ then \mathbf{u} is not feasible. We are done in this case. Now we assume for each i , $u_i > c_i$.

For each $S \subseteq T$, let $h(S) = v(S) - \sum_{i \in S} u_i$. Then h is a submodular function. A strongly polynomial time algorithm for the minimization of a submodular function was given in [FFI01, Sch00], hence we can find a set S^* which minimizes h in strongly polynomial time. It is easy to see, \mathbf{u} is feasible if and only if $h(S^*) \geq 0$.

In both the cases, if \mathbf{u} is infeasible, it is easy to see how to get a separating hyperplane. \square

The above theorem shows that for SNB games one can compute an (approximate) solution using the ellipsoid method. However, for SNB games, [Vaz09] shows that the solution is always rational and gives a faster algorithm. We sketch a continuous version of this algorithm since it will be crucial for proving many properties of SNB games and will be used in Sections 8 and 9.

Let $\mathbf{y} := \mathbf{u} - \mathbf{c}$. Then, an equivalent convex program is $\max\{\sum_i \alpha_i \log y_i : y(S) \leq f(S), \forall S \subseteq T; \mathbf{y} \geq \mathbf{0}\}$, where $f(S) := v(S) - c(S)$. Recall a set tight if $y(S) = f(S)$. Call an agent *free* if w.r.t the current \mathbf{y} it is not in any tight set.

An algorithm for SNB:

1. The algorithm maintains a set of tight sets \mathcal{T} which is initially empty. Also let $t = 0$.

2. while $T \notin \mathcal{T}$, increase t and let $y_i = \alpha_i \cdot t$ for all the free agents until some new set X gets tight. If X intersects with any set in \mathcal{T} , then since v is submodular, their union must be tight (Property 4.1). Pick X to be the maximal (inclusion-wise) tight set and put it in \mathcal{T} .

We have the following lemma:

Lemma 5.1.1. *The utility allocation returned by the above algorithm is an optimal solution to the convex program.*

Proof. The proof follows by giving dual values to the tight sets in \mathcal{T} which satisfy the KKT conditions. In particular, if one considers the above algorithm as a continuous time algorithm, then if a set X goes tight at time t (i.e. the value of y at which the set X went tight), it is priced $1/t$ and all the maximal sets of \mathcal{T} contained in X have their prices decreased by a factor $1/t$. Note that prices remain positive, and it can be easily checked that the KKT conditions hold. \square

6 Price of Bargaining

We recall the definition of the price of bargaining.

Definition 6.1. *For any valid function $v : 2^A \rightarrow R^+$, we define the price of bargaining of $F(v)$ to be $\min_{(\mathbf{c}, T) \in F(v)} \frac{u^*(\mathbf{c}, T)}{v(T)}$, where $u^*(\mathbf{c}, T) = \sum_i u_i^*(\mathbf{c}, T)$.*

Theorem 6.1. *For any valid function v , $F(v)$ has price of bargaining equal to 1 if and only if v is submodular.*

Proof. \Leftarrow : Suppose v is submodular. We want to show that for any disagreement point \mathbf{c} , and set $S \subseteq A$, if we restrict to the subproblem among agents in S , the Nash bargaining solution \mathbf{u}^* satisfies $\sum_{i \in S} u_i^* = v(S)$.

Since \mathbf{u}^* is the solution of Nash bargaining game, it must be Pareto optimal. Therefore every agent i is in some tight set T_i . Therefore by Property 4.1, we have $S = \cup_{i \in S} T_i$ is also tight, which means $\sum_{i \in S} u_i^* = v(S)$.

\Rightarrow : Suppose v is not submodular. By Property 4.2, there is a set $T = S \cup \{i, j\}$ and a feasible utility vector $\mathbf{u} = (u_k)_{k \in T}$ such that: (1) $u_k > 0, \forall k \in T$, (2) $S \cup i$ and $S \cup j$ are tight w.r.t. \mathbf{u} , (3) T is not tight w.r.t. \mathbf{u} . This is obtained from $\mathcal{F}_i \cap \mathcal{F}_j = \emptyset$.

Now for any $k \in T$, k is in some tight set w.r.t \mathbf{u} , hence by the lemma 6.1.1 below, there exists \mathbf{c} such that \mathbf{u} is the Nash bargaining solution corresponding to \mathbf{c} .

By condition 3 above, we have $\sum_{k \in T} u_k < v(T)$, which implies the Price of bargaining is strictly less than 1. \square

Lemma 6.1.1. *Given any valid function v , a set $T \subseteq A$ of agents, and a utility vector \mathbf{u} with $u_i > 0, \forall i \in T$, \mathbf{u} is Pareto optimal if and only if there exists a vector \mathbf{c} , with $c_i > 0 \forall i \in T$, such that \mathbf{u} is the bargaining solution for the instance (\mathbf{c}, T) .*

Proof. If \mathbf{u} is a bargaining solution it has to be Pareto optimal. We now prove the converse. Let $\delta := \min_{i \in T} u_i > 0$. If \mathbf{u} is Pareto optimal, we cannot increase u_i without changing other

coordinates of \mathbf{u} . Therefore, every agent i is in (at least) one tight set $Z_i \subseteq T$. Set the price of any tight set $Z_i \subseteq T$ to be $p_{Z_i} = P$, where $P > \frac{1}{\delta}$. Let c_i 's be defined as follows

$$c_i = u_i - \frac{1}{\sum_{Z \subseteq T: i \in Z} p_Z}, \quad (2)$$

We claim that the \mathbf{u}, \mathbf{c} and the prices satisfy the KKT conditions implying \mathbf{u} is the Nash bargaining solution with disagreement vector \mathbf{c} . This is because only tight sets are prices and c_i 's are so defined to satisfy the second KKT condition. It remains to check $c_i > 0$ for all i . This is because every agent has at least one tight set and therefore

$$c_i > u_i - \frac{1}{\frac{1}{\delta}} = u_i - \delta = 0.$$

□

7 Fairness

We begin by recalling the definition of fairness.

Definition 7.1. *Given any instance $I = (\mathbf{c}, T) \in F(v)$, let $\text{core}(I)$ be the set of all feasible Pareto optimal solutions. For any vector \mathbf{u} , let \mathbf{u}_{dec} (\mathbf{u}_{inc}) be the vector obtained by sorting the components of \mathbf{u} in decreasing (increasing) order. A vector \mathbf{x} min-max dominates (max-min dominates) \mathbf{y} if \mathbf{x}_{dec} (\mathbf{x}_{inc}) is lexicographically smaller than \mathbf{y}_{dec} (\mathbf{y}_{inc}).*

Definition 7.2. *Let \mathbf{u}^* be bargaining solution of instance $I = (\mathbf{c}, T)$. Instance I is said to be min-max fair (similarly max-min fair) if the vector $\mathbf{u}^* - \mathbf{c}$ min-max dominates (max-min dominates) $\mathbf{y} - \mathbf{c}$ for all $\mathbf{y} \in \text{core}(I)$. $F(v)$ is said to be min-max fair if all the instances in $F(v)$ are min-max fair.*

Theorem 7.1. *For any valid function v , $F(v)$ is min-max and max-min fair if and only if v is submodular.*

Proof. \Leftarrow : Suppose v is submodular. let \mathbf{u}^* be the Nash bargaining solution for (\mathbf{c}, T) where $T \subseteq A$. By corollary 4.1, we can choose the prices such that the tight sets w.r.t \mathbf{u}^* with positive price form a nested set family, $T = T_1 \supset T_2 \supset \dots \supset T_t \supset \emptyset$. Since for any agent i , $u_i^* - c_i = 1/(\sum_{j: i \in T_j} p_{T_j})$, we see that $(\mathbf{u}^* - \mathbf{c})_{\text{dec}}$ has elements of $T_1 \setminus T_2$ followed by those in $T_2 \setminus T_3$ and so on. Moreover, agents in $T_j \setminus T_{j+1}$ have the same $u_i - c_i$.

Pick any element $\mathbf{g} \neq \mathbf{u}^*$ in $\text{core}((\mathbf{c}, T))$. Suppose $\mathbf{g} - \mathbf{c}$ min-max dominates $\mathbf{u}^* - \mathbf{c}$. Since \mathbf{g} is Pareto optimal, every agent is in some tight set w.r.t \mathbf{g} . Hence by property 4.1, the whole set T is tight, i.e. $\sum_{k \in T} g_k = v(T)$. Since \mathbf{g} is feasible, we also have

$$\sum_{k \in T_2} g_k \leq v(T_2)$$

Since T_1 and T_2 are tight sets w.r.t \mathbf{u}^* , so we have

$$\sum_{k \in T_1 - T_2} g_k \geq \sum_{k \in T_1 - T_2} u_k^* \quad (3)$$

Since each agent i in $T_1 - T_2$ has the highest $u_i - c_i$ among all the agents, if $\mathbf{g} - \mathbf{c}$ min-max dominates $\mathbf{u}^* - \mathbf{c}$, then for any $k \in T_1 - T_2$, we have $g_k \leq u_k^*$. Then by (3), we have $g_k = u_k^*, \forall k \in T_1 - T_2$. Then we can use induction to show for any $1 \leq i \leq t$ and any $k \in T_i - T_{i+1}$, $g_k = u_k^*$. Hence $\mathbf{g} = \mathbf{u}^*$ which is a contradiction.

This proof also shows that $\mathbf{u}^* - \mathbf{c}$ is the unique min-max fair utility vector. By using an argument similar to [JV08], we can show that any unique min-max fair utility vector is also max-min fair.

\Rightarrow : Suppose v is not submodular, then by property 4.2, there is a set $T = S \cup \{i, j\}$ and a $\mathbf{g} = (g_k)_{k \in T}$ such that: (1) $g_k > 0, \forall k \in T$, (2) $S \cup \{i\}$ and $S \cup \{j\}$ are tight w.r.t \mathbf{g} , (3) $\mathcal{F}_l = \mathcal{F}_i \cup \mathcal{F}_j$, $\mathcal{F}_i \cap \mathcal{F}_j = \emptyset$.

For each $k \in T$, let $c_k = g_k - \epsilon$, where $0 < \epsilon < \min_{k \in T} \{g_k\}$. Clearly \mathbf{g} is a feasible core element corresponding to \mathbf{c} , since each k is in a tight set (either $S \cup \{i\}$ or $S \cup \{j\}$).

Let \mathbf{u}^* be the Nash bargaining solution corresponding to (\mathbf{c}, T) . Note that by definition, $\mathbf{g} - \mathbf{c}$ is a vector with all entries equal to ϵ . Thus, since \mathbf{g} is Pareto optimal, it is the unique min-max and max-min feasible solution. We now show that \mathbf{g} cannot be the solution to the Nash bargaining game implying the solution, \mathbf{u}^* is not fair.

If \mathbf{g} were the solution, by KKT conditions, we can price all the subsets of T such that:

$$\frac{1}{g_l - c_l} = \sum_{Z \in \mathcal{F}_l} p_Z = \sum_{Z \in \mathcal{F}_i} p_Z + \sum_{Z \in \mathcal{F}_j} p_Z = \frac{1}{g_i - c_i} + \frac{1}{g_j - c_j}$$

which contradicts the fact that $g_l - c_l = g_i - c_i = g_j - c_j = \epsilon$. □

8 Strong Monotonicity and Surplus Monotonicity

We recall the definition of strong monotonicity and surplus monotonicity.

Definition 8.1. For any valid function $v : 2^A \rightarrow R^+$, we say that $F(v)$ is strongly monotone if, for all games in $F(v)$, the following property holds: on increasing the disagreement utility c_i of an agent i , the bargaining solution doesn't increase the utility for any other agent j , where $j \neq i$.

Definition 8.2. For any valid function $v : 2^A \rightarrow R^+$, we say that $F(v)$ is surplus monotone if the surplus (i.e. $u_i^* - c_i$) of any agent i in the bargaining solution does not increase if her disagreement utility c_i increases. Formally, for any instance (\mathbf{c}, T) and (\mathbf{c}', T) , $T \subseteq A$, where $\mathbf{c}' = \mathbf{c}$ except that $c'_i > c_i$, then we have

$$u_i^*(\mathbf{c}', T) - c'_i \leq u_i^*(\mathbf{c}, T) - c_i.$$

In this section we show that any SNB game $F(v)$ is both strongly monotone and surplus monotone. Moreover, we also show that any UNB game which is strongly monotone must be an SNB game.

Theorem 8.1. For any submodular valid function v , $F(v)$ is surplus monotone and strongly monotone.

Proof. In this proof, we will use the algorithm from Section 5 for finding the optimal solution to the convex program (1) when the function v is a submodular function and refer the reader to it.

Suppose the disagreement of agent i goes from c_i to $c_i + \delta$. Call the new disagreement vector \mathbf{c}' . Let $f'(S) := v(S) - c'(S)$ for all S . To prove the theorem, it suffices to show that the optimum, \mathbf{y}' of the convex program $\max\{\sum_i \log y_i : y(S) \leq f'(S); \mathbf{y} \geq \mathbf{0}\}$ is dominated by \mathbf{y} , the solution to the original convex program with $f(\cdot)$. For this shows that $u'_i - c'_i \leq u_i - c_i$ implying surplus monotonicity and for $j \neq i$, we have $u'_j - c'_j \leq u_j - c_j$ and strong monotonicity follows from the fact that $c'_j = c_j$.

Claim 1. \mathbf{y} dominates \mathbf{y}' , i.e., $y'_j \leq y_j$, for all agents j .

Proof. We now use the continuous time algorithm of Section 5 to prove this. Firstly, note that $f'(S) = f(S)$ for all sets not containing i and $f'(S) = f(S) - \delta$ for all others. This implies, that there is at least one agent j with $y'_j < y_j$. Secondly, observe from the description of the algorithm that for any agent j with $y'_j < y_j$, there must be a corresponding tight set in \mathcal{T}' which contains both j and i .

We now show that if an agent j became non-free at time t in the original run (which means $y_j = t$), then by time t it must be in a tight set in the new run. We do so by showing that at time t if $y'_j = t$, then some set containing j at that time is tight (or over-tight which would imply $y'_j < t$).

Let A be the set containing j which went tight in the original run of the algorithm. Consider the set A in the new run of the algorithm at time t . Let $Q := \{k \in A : y'_k < y_k\}$. Note that if $j \in Q$, we are done. Assume $j \notin Q$. By the second observation made above and using the submodularity of v (to show union of intersecting tight sets is tight), we know there must exist a set Z which contains i and $Z \cap A = Q$, which is tight. That is, $y'(Z) = f'(Z) = f(Z) - \delta$. We claim that $y'(Z \cup A) \geq f'(Z \cup A)$ and thus we are done.

This is because

$$\begin{aligned} y'(Z \cup A) &= y'(A \setminus Q) + y'(Z) \geq y(A \setminus Q) + f'(Z) = y(A) - y(Q) + f'(Z) \\ &\geq f(A) - f(Q) + f(Z) - \delta \geq f(A \cup Z) - \delta = f'(A \cup Z) \end{aligned}$$

The first inequality follows from definition of Q , the second from the tightness of A under y and feasibility of y and the last follows from submodularity of f . □

□

Theorem 8.2. *If a valid function v is not submodular, then the UNB game $F(v)$ is not strongly monotone.*

Proof. Since v is not submodular, by property 4.2 there must exist a set S and agents $i, j \in A \setminus S$, $l \in S$, and a feasible utility vector \mathbf{u} such that: (1) $S \cup \{i\}$, $S \cup \{j\}$ are both tight w.r.t. \mathbf{u} , (2) $\mathcal{F}_l = \mathcal{F}_i \cup \mathcal{F}_j$, $\mathcal{F}_i \cap \mathcal{F}_j = \emptyset$, (3) $u_k > 0$, $\forall k \in T$, where $T = S \cup \{i, j\}$.

We will now construct an instance $(\mathbf{c}, T) \in F(v)$ which is not strongly monotone. Let $\delta = \min_{k \in T} u_k > 0$. For tight sets $S \cup \{i\}$, $S \cup \{j\}$, we set their prices to be $p_{S,i}$, $p_{S,j}$ respectively, where $p_{S,i} = p_{S,j} = P = \frac{2}{\delta}$. For any other set $Z \subseteq T$, we set its price p_Z to be zero.

Let

$$\forall k \in T \quad c_k = u_k - \frac{1}{\sum_{Z \subseteq T, k \in Z} p_Z}$$

Since $S \cup \{i\}$ and $S \cup \{j\}$ are both tight, so for any $k \in T$, there exist at least one $Z \subseteq S$ such that $p_Z = P$, and we have

$$c_k \geq u_k - \frac{\delta}{2} > 0.$$

By the definition of \mathbf{c} , all the KKT conditions hold, thus \mathbf{u} is the bargaining solution w.r.t. (\mathbf{c}, T) .

Suppose there exists a \mathbf{c}' and a corresponding bargaining solution \mathbf{u}' , such that: (1) $\forall k \in T, k \neq j, c'_k \geq c_k$, and (2) $c'_j = c_j$ and $u'_j > u_j$.

Using this, we can show that there exists a game in $F(v)$ which is not strongly monotone. This is because \mathbf{c}' can be obtained from \mathbf{c} by increasing only the coordinates other than j . If $F(v)$ is strongly monotone, then each time a coordinate of \mathbf{c} is increased utility allocated to j shouldn't increase. But if $u'_j > u_j$ is true then we get a contradiction.

Let \mathbf{u}' be same as \mathbf{u} except that $u'_j = u_j + \epsilon$, $u'_i = u_i + \epsilon$, $u'_l = u_l - \epsilon$. Using arguments similar to the proof of property 4.2, one can show that there exists small enough ϵ (given below) so that \mathbf{u}' is feasible.

$$\epsilon < \min\{\epsilon_0, u_l/2\}, \text{ where } \epsilon_0 := \min_{\text{non-tight } Z \subseteq T} \frac{(v(Z) - \sum_{k \in Z} u_k)}{2}.$$

Now we will construct \mathbf{c}' satisfying the above mentioned conditions, and *price* the sets so that along with \mathbf{u}' they satisfy the KKT conditions, which would imply that \mathbf{u}' is the bargaining solution for the disagreement point \mathbf{c}' .

We assign positive price to sets $S \cup i$, $S \cup j$ only, say $p'_{S,i}$ and $p'_{S,j}$ respectively, which is consistent with second KKT condition as both the sets are tight under \mathbf{u}' . Third KKT condition says that: 1) $c'_i = u'_i - \frac{1}{p'_{S,i}}$, 2) $c'_j = u'_j - \frac{1}{p'_{S,j}}$, and 3) $c'_k = u'_k - \frac{1}{p'_{S,i} + p'_{S,j}}$, $\forall k \in T, k \neq i, j$. Thus to get \mathbf{c}' , one can equivalently find prices $p'_{S,i}$ and $p'_{S,j}$.

Now, since we want $c'_j = c_j$, we get that

$$u'_j - \frac{1}{p'_{S,j}} = u_j + \epsilon - \frac{1}{p'_{S,j}} \Rightarrow \epsilon - \frac{1}{p'_{S,j}} = -\frac{1}{p_{S,j}}$$

Similarly, for $k \neq j$, we want $c'_k \geq c_k$. Expanding c'_k and c_k for different k , we get that following necessary condition should hold

$$-\epsilon - \frac{1}{p'_{S,i} + p'_{S,j}} \geq -\frac{1}{p_{S,i} + p_{S,j}}$$

It is not difficult so see that one can find $p'_{S,i}$ and $p'_{S,j}$ which satisfies above two conditions as long as the ϵ is chosen such that $\epsilon < \frac{1}{p_{S,i} + p_{S,j}} = \frac{\delta}{4}$.

To sum up, by setting $\epsilon = \min\{\epsilon_0/2, \delta/8\}$, we can find $p'_{S,i}, p'_{S,j}$ such that:

$$p'_{S,j} = \frac{1}{\epsilon + \frac{1}{p_{S,j}}}, p'_{S,i} \geq \frac{1}{\frac{1}{p_{S,i} + p_{S,j}} - \epsilon} - p'_{S,j}$$

Note that this value of ϵ is consistent with the previous mentioned upper bound on it. Therefore, we can construct \mathbf{c}' such that \mathbf{u}' is the bargaining solution w.r.t. \mathbf{c}' and $c'_k \geq c_k, \forall k \in T, c'_j = c_j$. Thus $(\mathbf{c}, T) \in F(v)$ is not strongly monotone. \square

However, SNB games are not the only UNB games which are surplus monotone. The following is an example of a UNB game which is not an SNB game but is still surplus monotone. It is an interesting open question to characterize surplus monotone games based on their valuation functions.

Example 1. $A = \{1, 2, 3\}$, $v : 2^A \rightarrow R^+$ is defined as: $v(\emptyset) = 0$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = 3$, $v(\{1, 2\}) = v(\{2, 3\}) = v(\{3, 1\}) = 4$, $v(\{1, 2, 3\}) = 6$.

9 Population Monotonicity

In this section, we investigate the population monotonicity in a UNB game. Generally speaking, population monotonicity means that when bargaining with a superset of agents, one cannot obtain more utility. This seems to be a reasonable property of bargaining problem, however, it turns out that the population monotonicity only holds in SNB games.

Definition 9.1. For any valid function $v : 2^A \rightarrow R^+$, we say that $F(v)$ is population monotone if, for any $T_1 \subset T_2 \subseteq A$ and any agent $i \in T_1$, agent i cannot obtain more utility in the bargaining solution of instance (\mathbf{c}, T_2) than in the bargaining solution of (\mathbf{c}, T_1) , i.e. $u_i^*(\mathbf{c}, T_1) \geq u_i^*(\mathbf{c}, T_2)$.

Theorem 9.1. For any valid function v , $F(v)$ is population monotone if and only if v is submodular.

Proof. \Leftarrow : The proof of this is similar to the proof that SNB games are strongly monotone. Once again, we use the continuous time algorithm of Section 5 to prove that for submodular v , $F(v)$ is population monotone and urge the readers to recall the same.

Consider the run of the algorithm when the set of agents is T_1 and T_2 . Let the utility vectors obtained be \mathbf{u}^1 and \mathbf{u}^2 respectively and let $\mathbf{y}^1 := \mathbf{u}^1 - \mathbf{c}(T_1)$ and $\mathbf{y}^2 := \mathbf{u}^2 - \mathbf{c}(T_2)$. It is enough to show that for all agents in T_1 , $y_i^1 \geq y_i^2$. Pick an agent $i \in T_1$ and suppose in the run of the algorithm with T_1 , at time t , i becomes non-free – that is, $y_i^1 = t$. We now show that in the run of the algorithm with T_2 , there exists a subset of T_2 containing i which is tight by time t (or over tight, which would imply $y_i^2 < t = y_i^1$).

Let A be the first set containing i which goes tight in the run of the algorithm with T_1 . Let $Q \subseteq A := \{j \in A : y_j^2 < y_j^1\}$. We may assume that $i \notin Q$ for otherwise we are done. Moreover, for any $j \in Q$, $y_j^2 < y_j^1$ implies that there must be a set $Z_j \subseteq T_2$ which goes tight in the run of the algorithm with T_2 and Z_j is not a subset of T_1 . Let Z be the union of all these sets $\{Z_j : j \in Q\}$. Note that $Q = Z \cap A$ and that Z is tight.

Consider the set $Z \cup A$ – we have

$$\begin{aligned} y^2(Z \cup A) &= y^2(Z) + y^2(A \setminus Q) \geq f(Z) + y^1(A \setminus Q) \\ &= f(Z) + y^1(A) - y^1(Q) \geq f(Z) + f(A) - f(Q) \geq f(Z \cup A) \end{aligned}$$

where the last but one inequality follows from the fact that A was tight w.r.t. \mathbf{y}^1 and \mathbf{y}^1 was feasible, and the last follows from submodularity of f .

\Rightarrow : If v is not submodular, by property 4.2, there exists a set $S \subseteq A$, $l \in S$, $i, j \in A \setminus S$ and a feasible utility vector \mathbf{u} such that the following hold:

1. $S \cup \{i\}$ and $S \cup \{j\}$ are both tight w.r.t \mathbf{u} .
2. $\mathcal{F}_l = \mathcal{F}_i \cup \mathcal{F}_j$ and $\mathcal{F}_i \cap \mathcal{F}_j = \emptyset$.
3. v is submodular on $S \cup \{i\}$.
4. for any $k \in T$, $u_k > 0$, where $T = S \cup \{i, j\}$.

For this utility \mathbf{u} , we can find \mathbf{c} such that \mathbf{u} is the bargaining solution of instance (\mathbf{c}, T) , i.e. $\mathbf{u} = \mathbf{u}^*(\mathbf{c}, T)$. We will show that after we remove agent j , there must be some agent in $S \cup \{i\}$ such that her utility in the bargaining solution of instance $(\mathbf{c}, S \cup \{i\})$ will be smaller.

Consider the instance $(\mathbf{c}, S \cup \{i\}) \in F(v)$, let the bargaining solution be $\mathbf{u}' = \mathbf{u}^*(\mathbf{c}, S \cup \{i\})$. Since v is submodular on $S \cup \{i\}$, we know that $S \cup \{i\}$ is tight w.r.t \mathbf{u}' . Recall that $S \cup \{i\}$ is also tight w.r.t \mathbf{u} , we have:

$$\sum_{k \in S \cup \{i\}} u_k = \sum_{k \in S \cup \{i\}} u'_k$$

If no agent's utility differs in two bargaining solutions, the set of tight sets does not change. Consider $r_l := \frac{1}{u_l - c_l}$, in the solution of instance (\mathbf{c}, T) :

$$r_l = \sum_{Z \in \mathcal{F}_l} p_Z = \sum_{Z \in \mathcal{F}_i} p_Z + \sum_{Z \in \mathcal{F}_j} p_Z = r_i + r_j$$

After we remove the agent j , since \mathcal{F}_j is removed from \mathcal{F}_l , i.e. $\mathcal{F}'_l = \mathcal{F}'_i$, thus we have $r_l = r_i$, which is a contradiction. So at least one agent's utility differs. Since

$$\sum_{k \in S \cup \{i\}} u_k = \sum_{k \in S \cup \{i\}} u'_k$$

there must be an agent $k \in S \cup \{i\}$ with $u'_k < u_k$.

Therefore, if any UNB game $F(v)$ is population monotone, v must be submodular. □

10 Extension to Nonsymmetric Nash Bargaining Games

Recall that $F(v, \alpha)$ is a family of nonsymmetric bargaining games, where function $v : 2^A \rightarrow R^+$ encodes the feasible set and $\alpha = (\alpha_i)_{i \in A}$ specifies the bargaining weights or clouts of all the agents.

Theorem 10.1. *For any valid function v and positive vector α , we have the following:*

1. $F(v, \alpha)$ has Price of Bargaining 1 iff v is submodular.
2. $F(v, \alpha)$ is strongly monotone iff v is submodular.
3. $F(v, \alpha)$ is max-min fair and min-max fair iff v is submodular.
4. $F(v, \alpha)$ is surplus monotone.
5. $F(v, \alpha)$ is population monotone w.r.t clouts iff v is submodular

Proof. All the proofs can be easily extended, and we only give key modifications here.

- (1) In equation (2), which is based on the KKT conditions, we do the following modification:

$$c_i = u_i - \frac{\alpha_i}{\sum_{Z \subseteq T: i \in Z} P_Z},$$

so that the lemma 6.1.1 still holds for the nonsymmetric case. Rest of the proof follows in similar fashion.

(3) For the case of fairness, we need to redefine the notion of fairness taking into account the relative bargaining powers of different agents. A bargaining game is said to be *min-max* (or *max-min*) fair if the vector $\langle \frac{u_i^* - c_i}{\alpha_i} \rangle$ dominates $\langle \frac{x_i - c_i}{\alpha_i} \rangle$ for all vectors \mathbf{x} in the core. Proof follows on the same lines as the one given in section 7

(5) The proof for one direction, namely $F(v, \alpha)$ is *population monotone w.r.t. clouts* when v is submodular, follows from the algorithm given in section 5 using the same arguments as used in section 8. For the other direction, we use the same examples as given in [CD07] and keep the disagreement vector to be the $\mathbf{0}$ vector. □

11 Open Problems

We believe this paper has only started the process of understanding game-theoretic properties of Nash and nonsymmetric bargaining games. Many of our criteria, e.g., for efficiency or fairness, are the most stringent possible, i.e., characterizing games having full efficiency or max-min and min-max fairness. There is probably much to be gained by considering intermediate notions of these properties.

A more specific problem is that we have proved surplus monotonicity for games in SNB and have given examples to show that this does not characterize SNB within UNB. We leave the open problem of characterizing the family of games within UNB that are surplus monotonic.

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