Low Rank Approximation of a Hankel Matrix by Structured Total Least Norm

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Abstract.

The structure preserving rank reduction problem arises in many important applications. The singular value decomposition (SVD), while giving the closest low rank approximation to a given matrix in matrix L_2 norm and Frobenius norm, may not be appropriate for these applications since it does not preserve the given structure.

We present a new method for structure preserving low rank approximation of a matrix, which is based on Structured Total Least Norm (STLN). The STLN is an efficient method for obtaining an approximate solution to an overdetermined linear system $AX \approx B$, preserving the given linear structure in the perturbation $[E \;\; F]$ such that $(A+E)X = B+F$. The approximate solution can be obtained to minimize the perturbation $[E \;\; F]$ in the L_p norm, where $p = 1, 2,$ or ∞. An algorithm is described for Hankel structure preserving low rank approximation using STLN with L_p norm. Computational results are presented, which show performances of the STLN based method for L_1 and L_2 norms for reduced rank approximation for Hankel matrices.

Key words: overdetermined linear systems, rank reduction, Hankel structure, Toeplitz structure, structured total least norm, total least squares, 1-norm, 2-norm, singular value decomposition.

1 Introduction

An important practical tool in many modern engineering and science applications is the singular value decomposition (SVD) [12]. For any matrix $Z \in \mathbf{C}^{m\times n}$, there exist unitary matrices

$$U = [u_1, \ldots, u_m] \in \mathbf{C}^{m\times m} \qquad \text{and} \qquad V = [v_1, \ldots, v_n] \in \mathbf{C}^{n\times n}$$

such that

$$Z = U\Sigma V^H \tag{1.1}$$

where $\Sigma = diag(\sigma_1, \ldots, \sigma_q) \in \mathbf{R}^{m\times n}$, with $q = min(m, n)$, is a diagonal matrix and $\sigma_1 \geq \cdots \geq \sigma_q \geq 0$. Throughout this paper, we will consider only the case when $m \geq n$, i.e., $q = n$. This SVD can also be represented as

$$Z = U \Sigma V^H = \sum_{i=1}^{n} \sigma_i u_i v_i^H. \tag{1.2}$$

It is well known that the SVD gives the best low rank approximation $\hat{Z}$, to a given matrix Z, which satisfies

$$\min \|Z - \hat{Z}\|_p, \quad rank(\hat{Z}) \leq k, \tag{1.3}$$

for any number $k \leq n$, where $p = 2$ or F. The solution is given by taking only the first k terms in (1.2) as

$$\hat{Z} = \sum_{i=1}^{k} \sigma_i u_i v_i^H. \tag{1.4}$$

A problem which is closely related to (1.3) is the Total Least Squares (TLS) problem [24] and it was originally discussed in [11] by Golub and Van Loan. For an overdetermined system

$$Ax \approx b \tag{1.5}$$

where $A \in \mathbf{C}^{m \times l}$ with $m > l$ and $b \in \mathbf{C}^{m \times 1}$, the TLS gives the solution that satisfies

$$\min_{E,x} \|[E \ r]\|_F \qquad \text{where} \qquad r = (A + E)x - b. \tag{1.6}$$

The matrix $[A + E \ \ b + r]$ is rank deficient since $b + r \in Range(A + E)$. It is well known that this minimum perturbation $[E \ r]$ for $[A \ b]$ can be represented as

$$[E \ r] = -[A \ b] v_{l+1} v_{l+1}^H = -\sigma_{l+1} u_{l+1} v_{l+1}^H$$

when $Z \equiv [A \ b]$ has its SVD

$$Z = \sum_{i=1}^{l+1} \sigma_i u_i v_i^H.$$

The above shows the relation between low rank approximation and the TLS solution since the rank deficient matrix

$$[A + E \ \ b + r] = \sum_{i=1}^{l} \sigma_i u_i v_i^H \in \mathbf{C}^{m \times (l+1)}$$

is the same as the best low rank approximant $\hat{Z}$ in (1.3), with $k = l$, for the matrix $[A \ b]$. Note that

$$\|[E \ r]\|_F = \sigma_{l+1}. \tag{1.7}$$

In many applications the data matrix Z has a special structure, such as Toeplitz, Hankel or sparse. For these applications, the SVD based solution for low rank approximation may not be appropriate, since it does not preserve the given structure. In fact, the solution (1.4) based on the SVD will typically be dense with no special structure, even when A is sparse or has a structure.

Recently, a new problem formulation, called Structured Total Least Norm (STLN), and an algorithm for computing the STLN solution have been developed [16, 17]. The STLN method preserves any given linear structure in E or $[E\ r]$ and can minimize these structured perturbations in the discrete L_p norm, where $p = 1, 2$ *or* ∞ [17]. A theoretical justification and computational testing of the STLN algorithm confirm that it is an efficient method for solving structured problems or when errors can occur only in some of the elements of A or $[A\ \ b]$ [17].

In this paper, we present a new method for structure preserving rank reduction, formulating the structure preserving low rank approximation problem as a constrained optimization problem. The method takes advantage of the equivalence between the STLN that preserves a given linear structure in the perturbation $[E\ r]$ and the low rank structured approximation. When there is no structure, an analogous equivalence can also be seen between the low rank approximation via SVD and the TLS solution for overdetermined systems, which is discussed above.

Other structure preserving methods for solving overdetermined systems such as STLS [8] and CTLS [1] can also be applied to the structure preserving rank reduction problem. However, these methods can produce solutions with p = 2 only and reduce the rank of the given matrix by one only. Another method which was developed specifically for structure preserving low rank approximation is due to Cadzow [4]. The performance of Cadzow's method is compared to that of the STLN based method for a certain rank reduction problem in Section 4. Also see [5] for two general procedures that have recently been developed by Chu, et al. for constructing the nearest approximation of a given matrix by one with a lower rank and any linear structure.

A classical problem where the new STLN based algorithm can have important impact is the model reduction problem with Hankel structure. This problem arises in many areas of engineering, such as speech encoding, filter design and high-order linear state-space models of dynamic systems, where reduced-order models can give significant improvement in solution techniques [25, 14]. In model reduction, given a Hankel matrix H, a Hankel matrix $\hat{H}$ of given rank k is to be found such that $\|H - \hat{H}\|_p$, for some matrix norm, is minimized. The STLN method for preserving Hankel structure in $[E\ \ r]$ can be easily modified to solve this problem for various norms. Moreover, STLN can be further extended to compute an approximant of any given rank k. It is straightforward to apply the results on Hankel matrices to Toeplitz matrices since the Toeplitz structure can be obtained simply by permuting the rows or columns of the Hankel matrix. In this paper, we will concentrate on the Hankel structure although the general idea presented in this paper applies to any linear structure. Our recent results on nonlinear extension of STLN [18] can also be applied to obtain *nonlinear* structured low rank approximation as well.

The organization of the paper is as follows. In Section 2, a STLN formulation and algorithm for solving an overdetermined system (1.5) with Hankel structured $[A\ b]$ and its extension to multiple right hand side vectors is presented. For L_1 norm minimization, we show how the formulation can be modified so that problems with complex data can be handled as linear program as well [17, 18]. Then in Section 3, it is shown that the STLN method that preserves Hankel structure in the perturbation $[E\ r]$ (Note that the structure is imposed not only on E but also on r) can be used to find a lower rank Hankel structured approximant. Computational test results which show the performance of the new algorithms are in Section 4. Finally, difficulties involved with structure preserving low rank approximation are discussed.

2 STLN for Overdetermined Systems with Hankel Structure

In STLN for solving the overdetermined system

$$AX \approx B \tag{2.1}$$

where $A \in \mathbf{C}^{m\times l}$ and $B \in \mathbf{C}^{m\times d}$, and $[A\ B]$ has Hankel structure, we are to find a minimum Hankel structured perturbation $[E\ F]$ so that $B+F \in Range(A+E)$.

First, we show that when all the antidiagonals in $[A\ \ B]$ are allowed to be perturbed, it is always possible to find a Hankel perturbation $[E\ \ F]$, so that $B + F \in Range(A + E)$.

THEOREM 2.1. *For any given integers m, n, and k with $k \le \min\{m,n\}$, there exists a Hankel matrix $H \in \mathbf{C}^{m\times n}$ such that $rank(H) = k$.*

The two methods presented in Section 4.1 for generating a Hankel matrix of any given rank can serve as the proofs of this theorem. ■

COROLLARY 2.2. *If every anti-diagonal in a Hankel matrix $[A\ \ B] \in \mathbf{C}^{m\times(l+d)}$, where $A \in \mathbf{C}^{m\times l}$ and $B \in \mathbf{C}^{m\times d}$, is allowed to be perturbed, then it is always possible to find a Hankel perturbation matrix $[E\ \ F] \in \mathbf{C}^{m\times(l+d)}$ so that $B + F \in Range(A + E)$.* ■

REMARK 2.1. *If some anti-diagonals in a Hankel structured matrix should be fixed, the above results may not be true. Consider the example*

$$H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Suppose the top three anti-diagonals are not allowed to be perturbed, then it is not possible to perturb the rest of H such that the third column lies in the range space of the first two columns. This example shows that the feasibility depends on the structure and perturbation pattern. ■

The following discussion on the formulation on STLN explains how we determine a minimum Hankel perturbation matrix $[E\ F]$, when it exists. Also we show how the formulation changes when we use the L_1 norm to handle complex valued data, so that linear programming techniques can still be used for L_1 norm minimization with complex data. One of the advantages of STLN formulation is that structured problems can be solved using different norms. In particular, when L_1 norm is used, then the method is more robust in the presence of large errors in data, which we call "outliers" [17, 18]. For the general discussion of Structured Total Least Norm (STLN) that preserves the given structure in E or $[E\ F]$, see [17, 25].

2.1 Single Right Hand Side Vector

In this section, we present the STLN method for solving the overdetermined system

$$Ax \approx b \tag{2.2}$$

where $A \in \mathbf{C}^{m\times l}$ and $b \in \mathbf{C}^{m\times 1}$, and $[A\ b]$ has Hankel structure, and all the anti-diagonals in $[A\ b]$ are subject to error. The STLN solution will find a minimum $[E\ f] \in \mathbf{C}^{m\times(l+1)}$, which also has Hankel structure, so that

$$(A+E)x = b+f \tag{2.3}$$

for some vector x, i.e., $b+f \in Range(A+E)$.

In solving the problem (2.2) using STLN, a vector $\eta \in \mathbf{C}^{(m+l)\times 1}$ is used to represent the $m+l$ free elements (the anti-diagonals) in the Hankel perturbation matrix $[E\ f]$. Specifically, when

$$[E\ f] = \begin{pmatrix} \eta_1 & \eta_2 & \dots & \eta_{l+1} \\ \eta_2 & \eta_3 & \dots & \eta_{l+2} \\ \vdots & \vdots & \dots & \vdots \\ \eta_m & \eta_{m+1} & \dots & \eta_{l+m} \end{pmatrix},$$

the vector η is defined as

$$\eta = (\,\eta_1 \quad \eta_2 \quad \dots \quad \eta_{l+m}\,)^T.$$

Note that the perturbation E depends only on

$$\alpha \equiv (\,\eta_1 \quad \eta_2 \quad \dots \quad \eta_{l+m-1}\,)^T \tag{2.4}$$

and the perturbation on the right hand side is

$$f = (\,\eta_{l+1} \quad \eta_{l+2} \quad \dots \quad \eta_{l+m}\,)^T.$$

Therefore, the vectors α and f can be represented as $\alpha = P_0\eta$ and $f = P_1\eta$, where

$$P_0 = (\,I_{(l+m-1)\times(l+m-1)} \quad 0_{(l+m-1)\times 1}\,) \in \mathbf{R}^{(l+m-1)\times(l+m)},$$

and

$$P_1 = (\, 0_{m\times l} \quad I_{m\times m}\,) \in \mathbf{R}^{m\times(l+m)}.$$

The vector α and the Hankel matrix E are equivalent in the sense that given α, E is known, and vice versa.

Now, the *structured residual* $\hat{r}$ for the perturbation $[E\ f]$ is defined as

$$\hat{r} = \hat{r}(\eta, x) = b + f - (A + E)x.$$

As shown in Theorem 2.1, when $[A\ b]$ has Hankel structure, it is always possible to find the perturbation $[E\ f]$ with Hankel structure such that $b{+}f \in Range(A{+}E)$. Therefore, we can force $\hat{r}$ to be zero to obtain the Hankel structured solution $[A{+}E\ \ b{+}f]$. Accordingly, the problem is formulated as the following constrained minimization problem [2, 26],

$$\begin{array}{rl} \min\limits_{\eta,x} & \|D\eta\|_p\,, \\ \text{subject to} & \hat{r} = 0. \end{array} \tag{2.5}$$

where D is a diagonal matrix of positive weights that accounts for the repetition of each η_i. For $p = 2$, this is a constrained minimization problem that could be solved by a general nonlinear programming code, such as MINOS [15]. The equivalent objective function $\|D\eta\|_2^2$, is quadratic and convex, but the constraints $\hat{r} = 0$, are a set of m indefinite quadratic equality constraints, so that this would be a difficult problem for such a code. Furthermore, a different (more difficult) formulation would be required for $p = 1$ or $p = \infty$.

The method of solution to be described now is more direct, and simpler, since it takes full advantage of the problem structure. Also it has the additional advantage that the same formulation and algorithm can be used for all 3 norms, L_p with $p = 1, 2$, and ∞. We solve it by using the penalty method [2], which transforms (2.5) into

$$\min_{\eta,x} \left\| \begin{pmatrix} \omega\hat{r}(\eta,x) \\ D\eta \end{pmatrix} \right\|_p, \tag{2.6}$$

where ω is a large penalty value. It is shown in [2] that when the penalty method with $p = 2$ is used for solving equality constrained least squares problems, Givens rotation based algorithms produce accurate results, regardless of row sorting and even with extremely large penalty values. For theoretical analysis of asymptotic behavior of the penalty method, see [22]. After the penalty method is applied, it is solved by using a linear approximation to $\hat{r}(\eta, x)$. Let $\Delta\eta$ and Δx represent a small change in η and x, respectively, and ΔE represent the corresponding change in E. Then the first order approximation to $\hat{r}(\eta + \Delta\eta, x + \Delta x)$ is

$$\begin{aligned} \hat{r}(\eta + \Delta\eta, x + \Delta x) &= b + P_1(\eta + \Delta\eta) - (A + E + \Delta E)(x + \Delta x) \\ &\approx b + P_1\eta - (A + E)x + P_1\Delta\eta \\ &\quad -(A + E)\Delta x - \Delta E x. \end{aligned} \tag{2.7}$$

Introducing a matrix $Y \in \mathbf{C}^{m\times(m+l-1)}$ that satisfies

$$Ex = Y\alpha = YP_0\eta, \tag{2.8}$$

(2.6) becomes

$$\min_{\Delta x, \Delta\eta} \left\| \begin{pmatrix} \omega(YP_0 - P_1) & \omega(A+E) \\ D & 0 \end{pmatrix} \begin{pmatrix} \Delta\eta \\ \Delta x \end{pmatrix} + \begin{pmatrix} -\omega\hat{r} \\ D\eta \end{pmatrix} \right\|_p . \tag{2.9}$$

For details on how the matrix Y is formed, see [17]. The (1,1) block of the matrix

$$M \equiv \begin{pmatrix} \omega(YP_0 - P_1) & \omega(A+E) \\ D & 0 \end{pmatrix} \in \mathbf{C}^{(2m+l)\times(2l+m)}$$

has Toeplitz structure as shown in the following example.

EXAMPLE 2.1. *Suppose*

$$E = \begin{pmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_2 & \eta_3 & \eta_4 \\ \eta_3 & \eta_4 & \eta_5 \\ \eta_4 & \eta_5 & \eta_6 \end{pmatrix}, \quad \textit{and} \quad f = \begin{pmatrix} \eta_4 \\ \eta_5 \\ \eta_6 \\ \eta_7 \end{pmatrix}.$$

Then

$$D^2 = diag(1,2,3,4,3,2,1), \qquad Y = \begin{pmatrix} x_1 & x_2 & x_3 & 0 & 0 & 0 \\ 0 & x_1 & x_2 & x_3 & 0 & 0 \\ 0 & 0 & x_1 & x_2 & x_3 & 0 \\ 0 & 0 & 0 & x_1 & x_2 & x_3 \end{pmatrix}$$

and $YP_0 = (Y\ 0) \in \mathbf{C}^{m\times(m+l)}$ *is also Toeplitz.* ■

The above discussions are summarized in Algorithm H-STLN for Hankel structure preserving solution for (2.1). For a fast algorithm that takes advantage of the structure of the matrix M, see [16].

For $p = 1$ and $p = \infty$, Eqn (2.9) can be solved as a linear program when the matrix A and the vector b are real [17, 18]. However, when the data has complex values, we cannot use a linear program since the absolute value of a complex number involves computation of squares. Accordingly, we modify the formulation (2.6) into

$$\min_{\eta_R, \eta_I, x_R, x_I} \left\| \begin{pmatrix} \omega\hat{r}_R(\eta, x) \\ \omega\hat{r}_I(\eta, x) \\ D\eta_R \\ D\eta_I \end{pmatrix} \right\|_p , \tag{2.10}$$

where the subscripts R and I denote real and pure imaginary parts, respectively, e.g., $x = x_R + \sqrt{-1}x_I$. In fact, for $p = 2$, (2.10) is the same as (2.6) since

$$\left\| \begin{pmatrix} \omega(\hat{r}_R + \sqrt{-1}\hat{r}_I) \\ D(\eta_R + \sqrt{-1}\eta_I) \end{pmatrix} \right\|_2 = \left\| \begin{pmatrix} \omega\hat{r}_R \\ \omega\hat{r}_I \\ D\eta_R \\ D\eta_I \end{pmatrix} \right\|_2 . \tag{2.11}$$

Algorithm H-STLN

Input – A vector b, a matrix A, with Hankel structure in $[A \;\; b]$, and tolerance tol. $E = 0, f = 0$.
Output – Hankel matrix $[E \;\; f]$, structured residual $\hat{r}$, and vector x such that $(A+E)x = b + f - \hat{r}$. Upon convergence, $\hat{r} \approx 0$.

1. Compute x from $\min \|Ax - b\|_p$. Form Y from x. Set $\hat{r} = b - Ax$.
2. **repeat**

   (a) $\underset{\Delta x, \Delta\eta}{\text{minimize}} \left\| \begin{pmatrix} \omega(YP_0 - P_1) & \omega(A+E) \\ D & 0 \end{pmatrix} \begin{pmatrix} \Delta\eta \\ \Delta x \end{pmatrix} + \begin{pmatrix} -\omega\hat{r} \\ D\eta \end{pmatrix} \right\|_p .$

   (b) Set $x := x + \Delta x$, $\eta := \eta + \Delta\eta$.

   (c) Form E and f from η, form Y from x, $\hat{r} = b + f - (A+E)x$.

   until ($\|\Delta x\| \le tol$ and $\|\Delta\eta\| \le tol$)

After linearization, (2.10) becomes:

$$\min_{\substack{\Delta x_R, \;\; \Delta x_I, \\ \Delta\eta_R, \;\; \Delta\eta_I}} \left\| M \begin{pmatrix} \Delta\eta_R \\ \Delta\eta_I \\ \Delta x_R \\ \Delta x_I \end{pmatrix} + \begin{pmatrix} -\omega\hat{r}_R \\ -\omega\hat{r}_I \\ D\eta_R \\ D\eta_I \end{pmatrix} \right\|_p , \tag{2.12}$$

where

$$M = \begin{pmatrix} \omega(Y_R P_0 - P_1) & -\omega(Y_I P_0) & \omega(A_R + E_R) & -\omega(A_I + E_I) \\ \omega(Y_I P_0) & \omega(Y_R P_0 - P_1) & \omega(A_I + E_I) & \omega(A_R + E_R) \\ \\ D & 0 & 0 & 0 \\ 0 & D & 0 & 0 \end{pmatrix} .$$

The matrices Y_R and Y_I can be chosen as in (2.8) to satisfy

$$E_R x_R = Y_R \alpha_R, \; E_I x_R = Y_R \alpha_I, \; E_R x_I = Y_I \alpha_R, \; E_I x_I = Y_I \alpha_I .$$

With this formulation, since every entry is real, the $p = 1$ and $p = \infty$ cases can be solved as linear programs. For problems with real data, (2.12) becomes (2.9).

2.2 Multiple Right Hand Side Vectors

The STLN method presented in Section 2.1 for a single right hand side vector can be extended to handle multiple right hand sides (RHS) [25]. Suppose the following overdetermined system with d RHS is given:

$$AX \approx B, \tag{2.13}$$

where $[A \quad B]$ has Hankel structure, $A \in \mathbf{C}^{m\times l}$, and denoting the ith column of B as $B_{\cdot i}$, $B = [B_{\cdot 1} \ \ldots \ B_{\cdot d}] \in \mathbf{C}^{m\times d}$. The problem is to find the minimum $[E \quad F] \in \mathbf{C}^{m\times(l+d)}$, which also has Hankel structure, so that $(A+E)X = B+F$ for some matrix $X = [X_{\cdot 1} \ \ldots \ X_{\cdot d}] \in \mathbf{C}^{n\times d}$.

Assume the Hankel matrix $[E \quad F]$ is represented as

$$[E\ F] = \begin{pmatrix} \eta_1 & \ldots & \eta_l & \eta_{l+1} & \ldots & \eta_{l+d} \\ \eta_2 & \ldots & \eta_{l+1} & \eta_{l+2} & \ldots & \eta_{l+d+1} \\ \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\ \eta_m & \ldots & \eta_{l+m-1} & \eta_{l+m} & \ldots & \eta_{l+d+m-1} \end{pmatrix}.$$

As in the case of $d = 1$, a vector

$$\eta = (\, \eta_1 \quad \ldots \quad \eta_{m+l+d-1} \,)^T \in \mathbf{C}^{(m+l+d-1)\times 1}$$

is used to represent $m+l+d-1$ free elements in the Hankel perturbation matrix $[E \quad F]$. Then the perturbation E depends only on α which is defined in (2.4). In addition, the ith column $F_{\cdot i}$ of $F = [F_{\cdot 1} \ \cdots \ F_{\cdot d}]$ is

$$F_{\cdot i} = (\, \eta_{l+i} \quad \eta_{l+i+1} \ldots \quad \eta_{l+m+i-1} \,)^T .$$

Therefore, $\alpha = P_0\eta$ where

$$P_0 = (\, I_{(l+m-1)\times(l+m-1)} \quad 0_{(l+m-1)\times d} \,) \in \mathbf{R}^{(l+m-1)\times(l+m+d-1)}$$

and each vectors $F_{\cdot i}$, $1 \le i \le d$ can be represented as $F_{\cdot i} = P_i\eta$ where

$$P_i = (\, 0_{m\times(l+i-1)} \quad I_{m\times m} \quad 0_{m\times(d-i)} \,) \in \mathbf{R}^{m\times(l+m+d-1)}.$$

The structured residual matrix $\hat{R}$ is defined as

$$\hat{R}(\eta, X) = B + F - (A+E)X.$$

For any matrix $X = [X_{\cdot 1}\ X_{\cdot 2} \ \cdots \ X_{\cdot d}]$, define

$$vec(X) = \begin{pmatrix} X_{\cdot 1} \\ X_{\cdot 2} \\ \vdots \\ X_{\cdot d} \end{pmatrix}$$

by stacking the columns of X into a long vector. The key idea in formulating the multiple RHS problem as STLN lies in the fact that (2.13) can be written as the following equivalent structured overdetermined system with a single right hand side vector,

$$\text{(2.14)} \qquad diag(A, \ldots, A)vec(X)$$
$$\equiv \begin{pmatrix} A & 0 & \ldots & 0 \\ 0 & A & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & A \end{pmatrix}_{md\times nd} \begin{pmatrix} X_{\cdot 1} \\ X_{\cdot 2} \\ \vdots \\ X_{\cdot d} \end{pmatrix} \approx \begin{pmatrix} B_{\cdot 1} \\ B_{\cdot 2} \\ \vdots \\ B_{\cdot d} \end{pmatrix} \equiv vec(B).$$

When A is perturbed to $A+E$ and B is perturbed to $B+F$ we want $(A+E \;\; B+F)$ to be Hankel. Also the diagonal blocks in (2.14) can only be perturbed to be the same matrix. When (2.14) is solved, the solution matrix X will be found in the vectorized form $x = vec(X)$. As in the case of $d = 1$, which is shown in Section 2.1, the problem is now formulated as the following minimization problem,

$$\min \| \begin{pmatrix} \omega vec(\hat{R}(\eta, x)) \\ D\eta \end{pmatrix} \|_p. \tag{2.15}$$

In order to compute the STLN solution $x = vec(X)$, we need to construct an $md \times (m + l - 1)$ matrix $Y \in \mathbf{R}^{md \times (m+l-1)}$ such that

$$diag(E, \ldots, E)x = Y\alpha. \tag{2.16}$$

Letting $Y = [Y_1^T \; Y_2^T \; \ldots Y_d^T]^T$, where each $Y_i \in \mathbf{C}^{m \times (m+l-1)}$, (2.16) gives

$$EX_{.i} = Y_i \alpha,$$

for $1 \le i \le d$. Because the structure of Y is determined by the structure of E, and E is independent of the column index i of $X_{.i}$, all Y_i's have the same structure. The only difference is the values of the matrix elements in Y_i: Y_i consists of the n elements of $X_{.i}$, $1 \le i \le d$. For example, consider Example 2.1 in Section 2.1 and assume that there are two right hand side vectors. Then, we have

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} x_{11} & x_{21} & x_{31} & 0 & 0 & 0 \\ 0 & x_{11} & x_{21} & x_{31} & 0 & 0 \\ 0 & 0 & x_{11} & x_{21} & x_{31} & 0 \\ 0 & 0 & 0 & x_{11} & x_{21} & x_{31} \\ x_{12} & x_{22} & x_{32} & 0 & 0 & 0 \\ 0 & x_{12} & x_{22} & x_{32} & 0 & 0 \\ 0 & 0 & x_{12} & x_{22} & x_{32} & 0 \\ 0 & 0 & 0 & x_{12} & x_{22} & x_{32} \end{pmatrix}, \tag{2.17}$$

where

$$X = [X_{.1} \; X_{.2}] = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{pmatrix}.$$

Now, the vectorized form of the structured residual $\hat{R}$ is

$$\hat{r}(\eta, X) = vec(\hat{R}(\eta, X)) = vec(B + F) - diag(A + E, \ldots, A + E)vec(X).$$

For small changes $\Delta\eta$ and $vec(\Delta X)$ in η and X, respectively, the first order approximation to $\hat{r}(\eta, X)$ is

$$\begin{aligned} & \hat{r}(\eta + \Delta\eta, X + \Delta X) \\ = \;\; & vec(B) + \begin{pmatrix} P_1 \\ \vdots \\ P_d \end{pmatrix} (\eta + \Delta\eta) \end{aligned}$$

$$
\begin{aligned}
& -diag(A+E+\Delta E,\ldots,A+E+\Delta E)vec(X+\Delta X) \\
\approx\ & \hat{r}(\eta,X) + \begin{pmatrix} P_1 \\ \vdots \\ P_d \end{pmatrix} \Delta\eta - diag(A+E,\ldots,A+E)vec(\Delta X) \\
& -diag(\Delta E,\ldots,\Delta E)vec(X).
\end{aligned}
$$

Since

$$diag(\Delta E,\ldots,\Delta E)vec(X) = Y\Delta\alpha = YP_0\Delta\eta,$$

with the linearization, the problem (2.15) becomes

$$\min_{\Delta x,\Delta\eta} \left\| M \begin{pmatrix} \Delta x \\ \Delta\eta \end{pmatrix} + \begin{pmatrix} -\omega\hat{r} \\ D\eta \end{pmatrix} \right\|_p, \tag{2.18}$$

where

$$M = \begin{pmatrix} \omega diag(A+E,\ldots,A+E) & \omega(YP_0 - \begin{pmatrix} P_1 \\ \vdots \\ P_d \end{pmatrix}) \\ 0 & D \end{pmatrix}.$$

As shown in Section 2.1 for $d=1$, the formulation (2.15) needs to be modified if we want to solve it as a linear program when $[A\ \ B]$ is complex. We skip the details here since the reformulation is analogous to what is shown in Section 2.1.

3 Hankel Structure Preserving Rank Reduction

In this section, we show how the results presented in Section 2 can be applied to find the Hankel structure preserving low rank approximation. The key idea is based on the fact that the STLN solution that preserves the structure in the perturbation $[E\ f]$ for $[A\ b]$ gives the structure preserving reduced rank approximation.

The problem that we want to solve is to find a Hankel matrix $\hat{H} \in \mathbf{C}^{m\times n}$

$$
\begin{aligned}
& \min && \|H-\hat{H}\|_p \\
& \text{subject to} && rank(\hat{H}) \le k < n \text{ and } \hat{H} \text{ is Hankel,}
\end{aligned} \tag{3.1}
$$

where a Hankel matrix $H \in \mathbf{C}^{m\times n}$, $m>n$, with $rank(H)=n$, is given. Note that (3.1) is feasible according to Corollary 2.2.

For simplicity of discussion, we first assume that $k=n-1$ and that the matrix H is partitioned as

$$H = [A\ b\,] \tag{3.2}$$

where A consists of the first $n-1$ columns of H and b is the last column of H. If we solve the overdetermined system

$$Ax \approx b \tag{3.3}$$

using STLN method for Hankel structured $[A\ b]$ presented in Section 2.1, then we will obtain a minimum Hankel structured perturbation $[E\ f]$ such that

$$b + f \in Range(A + E).$$

Therefore, $[A+E\ \ b+f]$ is a solution that has Hankel structure and the property that $rank([A+E\ \ b+f]) \le n-1$. When $rank(H) = n$, any of the n columns of H can be chosen to be the right hand side vector b in obtaining the reduced rank approximant $\hat{H}$ by solving an overdetermined system with a structure preserving method such as STLN. Therefore, one main difference between STLN for solving the overdetermined system $Ax \approx b$ where $[A\ \ b]$ is Hankel, and using it for structure preserving low rank approximation is that in the latter case, there are n choices for the right hand side vector b.

The choice of the column that becomes the right hand side vector does not change the *mathematical* solution of the minimization problem (3.1). However, the *computed* solution obtained from the STLN based method depends on the choice of the right hand side vector. This is because the different choice of the right hand side vector amounts to different initial values for the perturbation E and f in the STLN algorithm. Assume that a column, say the jth column, of H is chosen to be the right hand side vector b, i.e.,

$$H = [A_1\ b\ A_2] \tag{3.4}$$

where $A_1 \in \mathbf{C}^{m\times(j-1)}$, $b \in \mathbf{C}^{m\times 1}$ and $A_2 \in \mathbf{C}^{m\times(n-j)}$, $1 \le j \le n$, and consider the overdetermined system

$$[A_1\ \ A_2]x \approx b. \tag{3.5}$$

When the structure is not imposed, i.e., when the problem is solved as a TLS problem with the partitioning shown as in (3.4), it is clear that the choice of column from H to be the right hand side vector does not affect the solution. This is due to the fact that the SVD of $[A_1\ b\ A_2]$ gives the same expression (1.4) regardless of the value of j.

The *mathematical* solution of the structure preserving rank reduction problem, solved as an overdetermined system as in STLN, also does not depend on the choice of the column to be the right hand side vector, as explained in the following. Assume that an optimal solution for the problem (3.1) is $\hat{H}$ where H is partitioned as in (3.4). The perturbation $[E_1\ f\ E_2]$ with the Hankel structure, which is obtained from solving the overdetermined system (3.5) satisfies

$$[A_1 + E_1\ b + f\ A_2 + E_2] = \hat{H}$$

when the solution $\hat{H}$ is unique, regardless of the location of b. Note that $[E_1\ f\ E_2]$ is a minimum perturbation that satisfies

$$[A_1 + E_1\ A_2 + E_2]x = b + f,$$

for some vector x. Now, if we partition $\hat{E} \equiv \hat{H} - H$ as

$$\hat{H} - H = [\hat{E}_1\ \hat{f}\ \hat{E}_2],$$

where $\hat{E_1} \in \mathbf{C}^{m\times(j-1)}$, $\hat{f} \in \mathbf{C}^{m\times 1}$, $\hat{E_2} \in \mathbf{C}^{m\times(n-j)}$, it also satisfies the equation

$$[A_1 + \hat{E}_1 \; A_2 + \hat{E}_2]x = b + \hat{f},$$

for a vector x. This shows

$$\|H - \hat{H}\| \geq \|[E_1 \; f \; E_2]\|.$$

On the other hand, since $\hat{H}$ is a solution of (3.1), we have

$$\|H - \hat{H}\| \leq \|[E_1 \; f \; E_2]\|.$$

Therefore, even when the solution is not unique, the perturbation $[E_1 \; f \; E_2]$ satisfies the equality

$$\|H - [A_1 + E_1 \quad b + f \quad A_2 + E_2]\|_p = \|H - \hat{H}\|_p.$$

For multiple rank reduction, say rank reduction by $n - k > 1$, we can apply a similar procedure as for the case of rank 1 reduction ($k = n - 1$), which is described above. Suppose the matrix H is partitioned as

$$H = [A \quad B]$$

where A consists of the first k columns of H and B of the last $n - k$ columns of H. If we solve the overdetermined system

$$AX \approx B$$

using STLN method for Hankel structured $[A \quad B]$ presented in Section 2.2, then we will obtain a minimum Hankel perturbation $[E \quad F]$ such that

$$B + F \in Range(A + E).$$

Therefore, $[A + E \; B + F]$ is a solution that has Hankel structure and $rank([A + E \; B + F]) \leq k$ where $k < n - 1$. As in the case of $k = n - 1$, the choice of the right hand side will affect the *computed* solution obtained by the STLN based method. For more discussion on the choice of the columns of A to be the right hand side vectors, see Section 4.3.

4 Computational Results

The STLN based Hankel structure preserving rank reduction algorithm has been implemented in MATLAB with the unit roundoff of $\approx 10^{-16}$ in order to investigate its computational performance. Specifically, we present the computational results that illustrate the behavior of STLN algorithm with L_1 and L_2 norms for producing the rank reduced Hankel matrix. The test is performed for single rank reduction as well as rank reduction by 2. We also compare the results to those of Cadzow's algorithm [4]. The Cadzow's algorithm which was introduced in [4] uses composite property mapping that alternately finds the best (unstructured) lower rank approximation by the SVD and then restores the given structure by finding the closest matrix to the SVD approximation. The algorithm gives a lower rank structured matrix when it converges, but does not necessarily produces the *closest* structured lower rank matrix. Extra minimization steps need to be applied to obtain a better solution. This point was also observed in [7] and emphasized in [5].

For each of our tests, we constructed an $m \times n$ Hankel matrix H_c with $rank(H_c) = k < n$ for a specific value of k. Then the matrix H_c was perturbed to another Hankel matrix H so that $rank(H) = n$, which is used as the given data. In Section 4.1, we present two methods that can be used for constructing the Hankel matrix H_c with any given rank. The computational results are presented in Section 4.2. Finally, we present difficulties in structure preserving rank reduction, as well as in Cadzow's and STLN based algorithms, that we experienced, and discuss some ways to improve them.

4.1 Generating Hankel Matrices of a Given Rank

Unlike the unstructured cases, constructing a Hankel structured matrix of the given dimension and rank is not trivial. In this section, we present two methods for generating a Hankel matrix of any dimension $m \times n$ with $m \geq n$ and of a given rank $k \leq n$. These results are used in generating the test matrices in our computational tests.

The first method for generating a Hankel matrix of the given rank is well known and it is based on problems that occur in linear prediction [21, 20]. Suppose each of $m + n - 1$ data samples η_j can be represented as a sum of $k \leq n$ terms as

$$\eta_j = \sum_{l=1}^{k} \beta_l z_l^j, \quad \text{for } j = 1, 2, ..., m + n - 1, \tag{4.1}$$

where $\beta_l \neq 0,\ z_l \neq 0,\ l = 1, ..., k$, and $z_p \neq z_q$ for $p \neq q$. If we arrange these data in a Hankel matrix $H \in \mathbf{C}^{m \times n}$ as

$$H = \begin{pmatrix} \eta_1 & \eta_2 & \cdots & \eta_n \\ \eta_2 & \eta_3 & \cdots & \eta_{n+1} \\ \vdots & \vdots & \ldots & \vdots \\ \eta_m & \eta_{m+1} & \cdots & \eta_{m+n-1} \end{pmatrix},$$

then $rank(H) = k$. Also the jth column of H is a linear combination of the $(j-1)$th, $(j-2)$th, ..., $(j-k)$th columns of H, $j = k+1, ..., n$. [21, 20]. Notice that the rank of H is the same as the number of terms in the summation in (4.1). By using this method, we can construct either complex or real Hankel matrix of a given rank.

The second method is new and it is based on certain properties of circulant matrices [6]. Consider the $t \times t$ discrete Fourier transformation (DFT) matrix

F [27],

$$F = [f_1 \dots f_t] = \frac{1}{\sqrt{t}} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{t-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(t-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{t-1} & \omega^{2(t-1)} & \dots & \omega^{(t-1)(t-1)} \end{pmatrix}$$

where $\omega = exp(2\pi\sqrt{-1}/t)$. It is well known that the matrix F can diagonalize any circulant matrix of order t. In other words, any circulant matrix $C \in \mathbf{C}^{t\times t}$ can be represented as

$$C = F\Lambda F^* = \sum_{i=1}^{t} \lambda_i f_i f_i^* \tag{4.2}$$

where $\lambda_i,\ 1 \le i \le t$ are the eigenvalues of C and $\Lambda = diag(\lambda_1\, \lambda_2 \cdots \lambda_t)$.

On the other hand, any matrix $\tilde{C}$ that can be represented as

$$\tilde{C} = F\tilde{\Lambda}F^* = \sum_{i=1}^{t} \alpha_i f_i f_i^* \in \mathbf{C}^{t\times t} \tag{4.3}$$

for any $\alpha_i \in \mathbf{C}$, for $1 \le i \le t$, where $\tilde{\Lambda} = diag(\alpha_1\, \alpha_2 \cdots \alpha_t)$ is a circulant matrix. If exactly k of α_i's are non-zero, then $\tilde{C}$ has rank k. Moreover, the first k columns of $\tilde{C}$ are linearly independent. To show this, let P be the permutation matrix such that

$$P^*\tilde{\Lambda}P = \begin{pmatrix} \tilde{\Lambda}_1 & 0 \\ 0 & 0 \end{pmatrix},$$

where $\tilde{\Lambda}_1 \in \mathbf{C}^{k\times k}$ and $rank(\tilde{\Lambda}_1) = k$. Let $\tilde{F} = FP$. Then Eqn. (4.3) becomes

$$\tilde{C} = FPP^*\tilde{\Lambda}PP^*F^* = \tilde{F} \begin{pmatrix} \tilde{\Lambda}_1 & 0 \\ 0 & 0 \end{pmatrix} \tilde{F}^*.$$

It's sufficient to show that the submatrix G consisting of the first k rows and k columns of $\tilde{F}$ has full rank. The result follows since

$$G = \tilde{F}(1:k, 1:k) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \omega^{i_1} & \omega^{i_2} & \dots & \omega^{i_k} \\ \omega^{2i_1} & \omega^{2i_2} & \dots & \omega^{2i_k} \\ \vdots & \vdots & & \vdots \\ \omega^{i_1(k-1)} & \omega^{i_2(k-1)} & \dots & \omega^{i_k(k-1)} \end{pmatrix}$$

where $0 \le i_j \le t-1,\ j = 1, ..., k$ and $i_p \ne i_q$ when $p \ne q$. Therefore, when $t = m + n - 1$ for any positive integers m and n with $m \ge n$ and $k \le n$, the $m \times n$ submatrix T of $\tilde{C}$, which is defined as

$$T = \sum_{i=1}^{k} \alpha_i f_i(1:m) f_i^*(1:n) \in \mathbf{C}^{m\times n}$$

is Toeplitz and $rank(T) = k$. Moreover, the first k column of T are linearly independent. By permuting the rows of T by applying the anti-diagonal matrix I_m^A of order m from its left, we obtain the Hankel matrix

$$H = I_m^A T \tag{4.4}$$

with the property that $H \in \mathbf{C}^{m\times n}$, $rank(H) = rank(H(:, 1:k)) = k \leq n$.

In general, the Hankel matrix generated by the above method will have complex elements. However, using some properties of circulant matrices, we can generate a *real* Hankel matrix of a given rank. This is based on the fact that if α_i's in (4.3) are chosen such that

$$\alpha_1 \in \mathbf{R}, \quad \alpha_{i+1} = \overline{\alpha_{t-i+1}}, \quad i = 1, ..., t-1, \tag{4.5}$$

then $\tilde{C}$ is real. In fact, the (i,j)th element of $\tilde{C}$ is

$$\frac{1}{t}\sum_{l=0}^{t-1} \omega^{(i-j)l}\alpha_{l+1}.$$

Note that $\omega^{(i-j)l}$ and $\omega^{(i-j)(t-l)}$ are conjugate to each other and when (4.5) holds, the (i,j)th element of $\tilde{C}$ is real. Consequently, the Hankel matrix generated by the above method is also real.

4.2 Performance of the STLN based Rank Reduction Method

In general, the STLN based rank reduction algorithm will converge to the desired Hankel structured low rank approximation if the solution is close by, i.e., the given Hankel data matrix H of full rank is already close to the optimal Hankel matrix of desired lower rank. However, the practical question of how large the perturbation can be for convergence can only be answered by computational testing. For the STLN based methods, the last column of the matrix H was used as the right hand side vector b, and the penalty value of $\omega = 10^8$ was used.

We denote by STLN1 and STLN2, the STLN based algorithms with $p = 1$ and $p = 2$, respectively. For each test, a Hankel matrix $H_c \in \mathbf{R}^{7\times 5}$ with $rank(H_c) = k$ where $k = 3$ or $k = 4$ was generated by using either the linear prediction method or the circulant matrix method as described in Section 4.1. Then random perturbation vector $\delta \in \mathbf{R}^{11\times 1}$, where the components of δ are uniformly distributed random variables in [0, 1], was arranged to give a Hankel matrix $\Delta \in \mathbf{R}^{7\times 5}$. The perturbed Hankel matrix was generated as

$$H = H_c + \tau\Delta,$$

where τ is a constant which determined the level of perturbation in the given data matrix H. Then only the Hankel matrix H is assumed to be known. The objective is to test the accuracy and speed of the STLN1 and STLN2 based methods in recovering the actual solution H_c by reducing the rank of H preserving the Hankel structure.

For the purpose of checking the convergence, the methods were iterated until the smallest singular value of the computed matrix $\hat{H}$ is less than 10^{-14} for rank 1 reduction. The Cadzow's algorithm was also tested for the purpose of comparison. In each table, we show the total number of iterations for the STLN based and Cadzow's methods. Note that the major computational complexity of one iteration in the STLN1 and STLN2 based methods and the Cadzow's method requires a linear program (LP), a QR decomposition (QRD), and a SVD, respectively. Each of LP, QRD, and SVD can be computed by various algorithms, and the overall computational complexity depends on the specific implementation. Therefore, we only present the number of LP, QRD, and SVD's for each method. The maximum number of iterations of each method was set to be 20 LP's, 100 QRD's, and 100 SVD's, respectively. In Tables 1-4, the superscript "*" denotes the cases where the corresponding algorithm did not converge within these specified maximum number of iterations. The structured perturbation size $\|H - \hat{H}\|_F$ after the termination is also shown.

In Table 4.1, we present the test results of the following problem. In the test, the data generated from 11 samples η_j, were arranged in a 7×5 Hankel matrix H_c. Each data sample η_j is represented as a sum of 4 exponential terms as in (4.1) where

$$\beta = (1.0,\ 2.0,\ 0.5,\ 1.5) \qquad \text{and} \qquad z = (e^{-0.1},\ e^{-0.2},\ e^{-0.3},\ e^{-0.35}).$$

Therefore, $rank(H_c) = 4$. For each value of τ in Table 4.1, an average of the test results with 30 different random Hankel matrices Δ is shown. Note that for all values of τ, the smallest singular value became smaller than 10^{-14} in only several iterations in the STLN based algorithms when τ is less than 10^{-3}. The STLN1 based algorithm converged particularly fast. In Cadzow's algorithm, the smallest singular value became less than the value of τ after about 5 iterations. However, the convergence speed after the first several iterations was extremely slow, reducing the smallest singular value by a factor of 10^{-1} in every 10 to 15 more iterations, even when it converged.

The results presented in Table 4.2 are obtained with the same Hankel matrix $H_c \in \mathbf{R}^{7\times 5}$ as in the previous test. Then H_c was perturbed by a random Hankel matrix Δ, which is also generated as before, except that now an outlier of size 0.01 was added to η_8 in each case, and the table represents one test case for each value of τ. It is seen that for the STLN2 based algorithm and Cadzow's algorithm, the outlier dominated the convergence speed and the error in recovering H_c. In contrast, the STLN1 based algorithm produced a solution, which ignores the outlier and the accuracy of solution was influenced only by the size of τ. In particular, the error $\|H - \hat{H}\|_F$ with STLN1 is essentially proportional to τ only. This shows that the STLN1 based algorithm can be very robust in the presence of a large error in the data.

In the second test, we used the Hankel matrix H_c generated by the circulant matrix based method as described in Section 4.1 and solve the problem to reduce the rank by 2. The elements of the circulant matrix were randomly chosen from interval $[0,\ 1]$. The Hankel matrix $H_c \in \mathbf{R}^{7\times 5}$, with $rank(H_c) = 3$, was then

	# of iterations			$\|H-\hat{H}\|_F$		
τ	STLN1 (LP)	STLN2 (QRD)	Cadzow (SVD)	STLN1	STLN2	Cadzow
$e-8$	1.1	1.7	59.8	$3.419e-8$	$3.289e-8$	$3.289e-8$
$e-7$	1.6	2.3	75.3	$3.270e-7$	$3.120e-7$	$3.120e-7$
$e-6$	2.2	2.2	83.0	$3.501e-6$	$3.182e-6$	$3.182e-6$
$e-5$	2.1	3.2	92.4	$3.474e-5$	$3.366e-5$	$3.366e-5$
$e-4$	2.1	3.9	93.3	$3.461e-4$	$3.297e-4$	$3.297e-4$
$e-3$	4.0	6.8	100*	$3.730e-3$	$3.372e-3$	$3.373e-3$
$e-2$	4.5	20.5	100*	$3.482e-2$	$3.472e-2$	$3.496e-2$
$e-1$	6.9	22.6	100*	$3.330e-1$	$3.350e-1$	$3.401e-1$

Table 4.1: Rank 1 reduction of a Hankel matrix. Each entry is an average of 30 test results.

perturbed similarly to generate the full rank data matrix H as in the first test. The convergence criterion was that the 2 norm of the vector consisting of the smallest 2 singular values of the computed matrix $\hat{H}$ should be less than 10^{-14}. The results are shown in Table 4.3 and Table 4.4. For each value of τ in Table 4.3, an average of the test results with 30 different random Hankel matrices Δ is shown. The results presented in Table 4.3 show that all three methods produced good answers in this test. Table 4.4 shows the results obtained with the same Hankel matrix H_c as those in Table 4.3 except that now a large error of size 0.01 was added to η_9, and that the table represents one test case for each value of τ. The accuracy of solutions produced by all three methods was dependent on the largest perturbation, which is the outlier. The STLN1 based method

	# of iterations			$\|H-\hat{H}\|_F$		
τ	STLN1 (LP)	STLN2 (QRD)	Cadzow (SVD)	STLN1	STLN2	Cadzow
$e-8$	8	100*	95	$3.130e-8$	$1.806e-2$	$1.744e-2$
$e-7$	8	100*	100*	$4.052e-7$	$1.806e-2$	$1.744e-2$
$e-6$	8	100*	90	$4.864e-6$	$1.806e-2$	$1.745e-2$
$e-5$	8	100*	95	$3.764e-5$	$1.807e-2$	$1.745e-2$
$e-4$	8	100*	99	$2.742e-4$	$1.819e-2$	$1.756e-2$
$e-3$	6	100*	100*	$3.104e-3$	$1.952e-2$	$1.893e-2$
$e-2$	20	100*	100*	$4.637e-2$	$3.597e-2$	$3.502e-2$
$e-1$	10	100*	100*	$4.972e-1$	$4.180e-1$	$3.804e-1$

Table 4.2: Rank 1 reduction of a Hankel matrix with an outlier 0.01 at the 8th anti-diagonal

	# of iterations			$\|H - \hat{H}\|_F$		
τ	STLN1 (LP)	STLN2 (QRD)	Cadzow (SVD)	STLN1	STLN2	Cadzow
$e-8$	1.0	1.0	28.8	$4.212e-8$	$3.206e-8$	$3.178e-8$
$e-7$	2.5	1.3	33.4	$3.414e-7$	$3.192e-7$	$3.110e-7$
$e-6$	3.0	3.0	39.6	$3.368e-6$	$3.291e-6$	$3.291e-6$
$e-5$	3.2	3.5	44.9	$3.105e-5$	$3.139e-5$	$3.139e-5$
$e-4$	5.6	5.5	50.8	$3.364e-4$	$3.230e-4$	$3.230e-4$
$e-3$	6.7	7.3	38.7	$3.146e-3$	$3.084e-3$	$3.084e-3$
$e-2$	12.5	13.6	47.1	$3.211e-2$	$3.197e-2$	$3.197e-2$

Table 4.3: Rank 2 reduction of a Hankel matrix. Each entry is an average of 30 test results.

converged extremely fast especially when the noise level τ is low. However, the solutions produced by STLN1 are also affected by the outlier unlike the results presented in Table 4.2. This can be explained using our theoretical analysis: L_1 norm minimization is robust in the presence of outliers, but the solution may be affected by the outliers depending on the *locations* of the outliers [19].

We have conducted many other tests using different data matrices of various sizes, and similar results were obtained. Based on these implementation results, we conclude that the STLN based method is very efficient for obtaining low rank approximation with Hankel structure. The Cadzow's method often has difficulties with convergence and its computational complexity is high even when it converges. In all our implementations, E and f were initially set to be zero. For improving the speed and convergence of the STLN based methods, it will be important to find a good initial guess for them. The results from Cadzow's method can, in fact, give an initial guess. However, using the Cadzow's result after several iterations as the initial guess for the STLN based method did not necessarily improve the solution. Also Cadzow's method is too expensive even for providing an initial guess since each iteration involves an SVD.

4.3 Discussions

In this section, we discuss some difficulties involved with the problem of finding a structured low rank approximant, and also some difficulties with the algorithms we discussed in this paper for solving the problem. As shown in Section 2, the structured low rank approximation problem may not even be feasible. The feasibility depends on the structure which is imposed on the solution. For the Hankel structured problem where every anti-diagonal of the matrix is allowed to be perturbed, the feasible set is not empty as shown in Theorem 2.1. On the other hand, the difficulty with Hankel structured problem involves the fact that there may be infinitely many global solutions as illustrated in the following simple 2×2 example.

τ	# of iterations			$\|H-\hat{H}\|_F$		
	STLN1 (LP)	STLN2 (QRD)	Cadzow (SVD)	STLN1	STLN2	Cadzow
$e-8$	3	12	91	$2.127e-2$	$1.236e-2$	$1.234e-2$
$e-7$	3	12	91	$2.128e-2$	$1.236e-2$	$1.234e-2$
$e-6$	3	11	91	$2.132e-2$	$1.236e-2$	$1.234e-2$
$e-5$	3	14	91	$2.073e-2$	$1.236e-2$	$1.235e-2$
$e-4$	9	13	91	$4.615e-4$	$1.248e-2$	$1.247e-2$
$e-3$	9	19	91	$4.110e-3$	$1.293e-2$	$1.292e-2$
$e-2$	20*	52	90	$8.121e-1$	$3.904e-2$	$3.903e-2$

Table 4.4: Rank 2 reduction of a Hankel matrix with an outlier 0.01 at the 9th anti-diagonal

Consider the 2×2 anti-diagonal matrix

$$I_2^A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and the problem

$$\min \|I_2^A - T\|_F, \quad T \in \mathbf{R}^{2\times 2}, \quad rank(T) = 1 \tag{4.6}$$

where the solution T is required to have Toeplitz structure. It can be shown that (4.6) has infinitely many global solutions: the matrix

$$T = \begin{pmatrix} \frac{\mu}{\mu^2+1} & \frac{\mu^2}{\mu^2+1} \\ \frac{1}{\mu^2+1} & \frac{\mu}{\mu^2+1} \end{pmatrix}$$

is a solution for (4.6) for any $\mu \in \mathbf{R}$ and $\|T - I_2^A\|_F = 1$.

The next example shows that the choice of column which is to be the right hand side vector in the STLN based algorithm is not trivial. The example also illustrates that the permutation affects the solution in Cadzow's algorithm. Suppose the optimal Hankel approximant H_c with $rank(H_c) \le k$, for a Hankel matrix H has the minimum distance $\beta = \|H - H_c\|_F$. Then the distance between the permuted Hankel matrix $I^A H I^A$ and its best structure preserving reduced rank approximant also has the same value β since

$$\|I^A H I^A - I^A H_c I^A\|_F = \beta.$$

However, the following example shows that this is not necessarily true in both the STLN based algorithm and Cadzow's algorithm. Consider the rank 1 reduction problem for the Hankel matrix

$$H = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.7}$$

and its optimal Hankel approximant H_c with $rank(H_c) \leq 2$. The Cadzow's method gives the solution

$$\hat{H} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

after one iteration and further iteration does not change this solution. The STLN based method also gives the same solution in one iteration. It is easy to see that the computed solution $\hat{H}$ is an optimal solution since $\hat{H}$ is also the solution obtained by SVD without preserving the structure.

However, when these methods are applied to the permuted Hankel matrix

$$I_3^A H I_3^A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{4.8}$$

they produce poor results. In Cadzow's method, after the ith iteration of SVD and averaging along antidiagonals, we obtain

$$\hat{H}^{(i)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2^{-i} \\ 0 & 2^{-i} & 0 \end{pmatrix}.$$

Therefore, the computed answer converges to the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.9}$$

reducing the rank of H by 2 instead of 1, with $\|H - \hat{H}\|_F = \sqrt{2}$. The same problem is observed when our STLN based method was used: the computed solution was very close to the matrix (4.9) from the first iteration. In case of STLN based algorithms, the permutation in (4.8) amounts to choosing the first column, instead of the last column, of H in (4.7) to be the right hand side vector b. However, all three columns of H are orthogonal to each other and it is not clear how to determine the column which would be the best candidate to be the right hand side vector b. Moreover, the situation is reversed if we want rank 2 reduction. For rank 2 reduction, all three methods produced the correct answer $\hat{H}$ of (4.9) in one step when we start with the permuted matrix (4.8), while all methods had difficulty with the matrix H of Eqn. (4.7), producing a zero matrix, which is the result of rank 3 reduction. However, using the STLN1 method, we have some preliminary results which show that a suitable choice for the initial values of E and F will eliminate these difficulties.

In order to more fully develop the STLN based algorithm for structured low rank approximation problems, a number of areas need further study. Future work will include finding good initial values for STLN, choice of the column to use as the right hand side vector, and determining under what conditions STLN will give the minimum error solution when the errors in H are large.

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