1050 Midterm 2, March 30, 2006. NAME:__________________________. Grade:____ (______/1

Closed books. Individual test. Do not look at other people's work. Please write legibly. Use the back of sheet if needed.

/20 1) **Matrices**. Consider a coordinate system C_1 with origin O_1 and basis vectors I_1 and J_1. Let (x_1,y_1) represent the local coordinates of point P in C1. Provide a symbolic expression (using points and vectors) for P in the global coordinate system. *(3 points)* $P = O_1+x_1I_1+y_1J_1$

Let, $O_1=(5,2)$, $I_1=(0,1)$, $J_1=(-1,0)$, and $(x_1,y_1)=(3,2)$. Compute the numeric values for the global coordinates (x,y) of P. *(2 points)* $(x,y) = (3 , 5)$

Consider a 2nd coordinate system C_2 with origin O_2 and basis vectors I_2 and J_2. Let (x_2,y_2) be the local coordinates of P in C_2. Provide a symbolic expression (using points and vectors) for (x_2,y_2). *(3 points)* $(x_2,y_2) = (O_2P \bullet I_2 , O_2P \bullet J_2)$

Let, $O_2=(1,2)$, $I_2=(1,0)$, $J_2=(0,1)$. Compute the numeric values for (x_2,y_2). *(2 points)* $(x_2,y_2) = (2 , 3)$

Provide a homogeneous matrix M for mapping (x_1,y_1) into (x_2,y_2). *(4 points)*

$$\begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 4 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix}$$

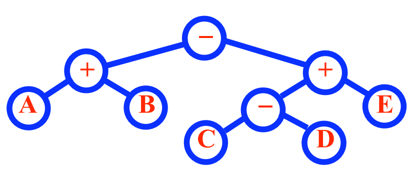
Verify that $(x_2,y_2,1)=M\,(x_1,y_1,1)$. *(1 point)* $(2 , 3) = (0\times3-1\times2+4 , 1\times3+0\times2+0)$.

Compute the square of M. *(4 points)*

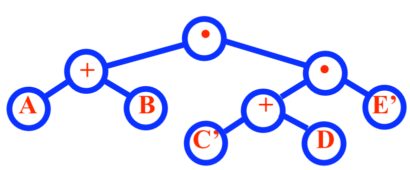
$$M^2 = \begin{pmatrix} -1 & 0 & 4 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

Is M^2 symmetric? *(1 point)* ___ No __

/15 2) **CSG**. Consider the CSG expression S=(A+B)–((C–D)+E). Draw the corresponding CSG tree. *(3 points)*



Draw its positive form (use X' to denote the complement of X). *(4 points)*



Write its disjunctive form *(4 points)*: AC'E'+BC'E'+ADE'+BDE'

Write the active zone of primitive D *(4 points)*: CE'(A+B)

/20 3) **Computational Geometry**. Consider a finite set of n sites (isolated points) P_i in the plane.

Triangle (P_i, P_j, P_k) is a Delaunay triangle when *(2 points)*: *no site P_i lies inside its circumscribing circle.*

Provide the pseudocode for the simple algorithm used in class for drawing the Delaunay triangulation. *(7 points)*

```
For each triplet (A,B,C) of different points {
        Compute the center X and radius r of circle though them;
        Found = False;
        For each other point D {If (XD•XD<r² ) found = True; };
        If (! found) {draw_triangle(A,B,C);
        }
```

What is the computational complexity of this algorithm *(1 points):* It is O(n^4)

The Voronoi region of site P_i is *(2 points)*: *the set of points closer to P_i than to any other site.*

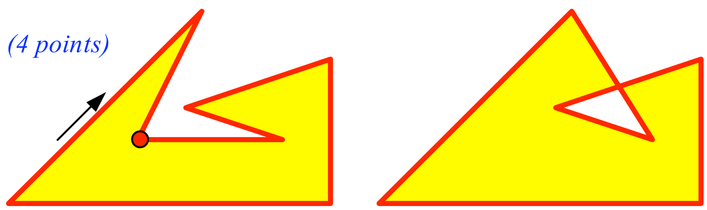
A set S is convex when *(2 points):* $\forall A \in S\ \forall A \in S$ $Line\_Segment(A,B) \subset S$

The convex hull H(P) of a polygon P is *(2 points)*: *the intersection of all convex sets that contain P.*

"The convex hull H(P) of a clockwise polygon P may be obtained by decimating all left-turn vertices."

Is this true? *No*

Justify your answer with a proof or counterexample. *(4 points)*



Removing the left-turn red-dot vertex from the polygon (left) will create a self-intersecting polygon with only right-turn vertices.

/15 4) **Proofs**. Use mathematical induction to prove that 3 divides $f(n)=n^3+2n$ for all positive integers n *(5 points)*:

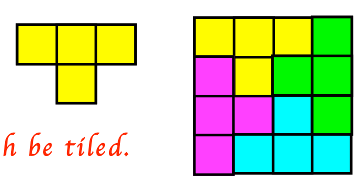
Basis step: P(1)=3, so it is true for n=1. Induction step: Assume that $f(n)= n^3+2n =3k$ for some integer k. $f(n+1)=(n+1)^3+2(n+1)=(n^3+3n^2+3n+1)+2(n+1)=(n^3+2n)+3n^2+3n+3=3(k+n^2+n+1)$, which is divisible by 3.

Prove that every $2^n \times 2^n$ (n > 1) chessboard can be tiled with T-ominos. *(5 points)*

Basis step: This is true for n=2 (see figure).

Induction: If it is true for n, then it is true for n+1

(by dividing the chessboard into four $2^n \times 2^n$ quadrants which can each be tiled.



Prove that any integer greater than 1 can be written as the product of primes. *(5 points)*

Basis step: P(2) is true since 2 is a prime.

Strong induction step: Assume P(j) is true for all integers j such that 1<j≤k. If k+1 is prime, then P(k+1) is true. Otherwise, k+1 can be written as the product AB of two integers such that 2≤a≤b<k+1. By strong induction, A and B are products of primes. Hence P(k+1).

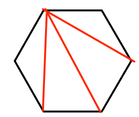
/30 5) **Counting**. How many different Boolean functions of n variables are there? Justify your answer *(3 points)*:

Each function is defined by a truth table of $t=2^n$ entries. There are 2^t different ways of assigning Boolean values to these entries, and hence 2^t different Boolean functions.

How many students should we have in a class to ensure that at least 3 have their birthday on the same month. *(2 points)*:

Pigeon-hole principle: $2\times 12+1 = 25$.

How many diagonals does an n-sided polygon have? *(3 points)*: $n(n-3)/2$.



Provide the next term in: 1, 7, 25, 79, 241, and the formula for term n *(3 points)*:

The next term is 243 and the formula is $3n-2$.

How many different sets of 5 coins can you make if you use only pennies, nickels, dimes, and quarters? *(4 points)*:

The sets are not ordered and have repetition, hence we use $C(n+r-1,r)=(n+r-1)!/(r!(n-1)!)$ with $n=4$ and $r=5$. $C(8,5)=8!/(5!3!)=8\times 7\times 6\times 5\times 4\times 3\times 2/(5\times 4\times 3\times 2\times 3\times 2)=8\times 7=56$.

Formulate the number C(n) of cells defined by an arrangement of n lines in the plane? Justify your derivation *(5 points)*:

The n^{th} line is split into n segments by the previous n−1 lines and hence adds n segments. Hence $C(0)=1$ and for $n>0$, $C(n)=C(n-1)+n$: $C(n)=1+1+2+3+\ldots+n=1+n(n+1)/2$

How many different 3-digit strings (including those starting with a 0) are there that either start with a 1 or 2 or end with an 8 or 9. Include explanations of how you derive your answer *(5 points)*:

There are 10^2 4-digit numbers starting with 1. Hence there are 2×10^2 numbers starting with a 1 or a 2. Similarly, there are 2×10^2 numbers ending with an 8 or a 9. If we add these, we double count the numbers that start with a 1 or 2 and end with an 8 or 9. There are 4×10^1 such numbers. Hence, the answer is: $4\times(10^2-10^1) = 4\times 9\times 10=360$.

How many positive integers not exceeding 100 are divisible by 4 or by 6? *(5 points)*:

25 are divisible by 4. 16 are divisible by 6. 8 are divisible by 6 and 4, i.e. by their $lcm(4,6)=12$. Hence the answer is $25+16-8=33$.