Smooth curves and surfaces are used for aesthetic, manufacturing, and analysis applications where discontinuities due to triangulated approximations would create misleading artifacts. I like to distinguish three classes of surfaces:
- implicit: \( f(x,y,z)=0 \), where \( f \) is often a polynomial of low degree (handy for computing intersections with rays)
- parametric surfaces: \( S(u,v) = (x(u,v),y(u,v),z(u,v)) \), where \( x, y, \) and \( z \) are often low degree polynomials in \( u \) and \( v \)
- generative surfaces, such as sweeps or subdivision surfaces, which are defined in terms of a construction procedure

Piecewise cubic parametric curves and surfaces are popular in CAD, animation, and graphics. A point \( C(t) \) on curve \( C \) has coordinates \( (x(t),y(t),z(t)) \), where \( x, y, \) and \( z \) are cubic polynomials in \( t \). The shape of \( C \) is defined by a control polygon with control points (i.e. vertices) \( P_i \). We discuss below how to subdivide the control polygon and how to evaluate \( C(t) \). To define a bi-cubic surface, express each \( P_i \) as a curve \( P_i(s) \). As \( s \) is varied, \( C(t) \) sweeps out a surface \( S(t,s) \).

1. **Split&tweak subdivision of control polygons a uniform cubic B-spline curves**
   Given a control polygon, for example \( a,b,c,d \), repeat the following sequence of two steps, until all consecutive 4-tuples of control points are nearly coplanar.
   1. Split: insert a new control point in the middle of each edge \( (2,4,6,8) \)
   2. Tweak: move the old control points half-way towards the average of their new neighbors \( (1,3,5,7) \)

The control polygon converges rapidly to the B-spline curve. This works whether the curve is closed or open.

2. **Converting a uniform cubic B spline into a series of cubic Bezier curves**
   Given a control polygon with vertices \( a,b,c, \ldots \) do: (1) insert new vertices \( w,2,3,5,\ldots \) to split each edge into 3 equal parts; (2) move the original vertices to the average of their immediate neighbors \( (b1,c4,\ldots) \); and (3) delete the first and last 3 vertices \( (a,w,x,y,z,i) \). The consecutive trigons, \( (1,2,3,4), (4,5,6,7), (7,8,9,10) \ldots \) are the control polygons of Bezier curves.

3. **Subdividing a cubic Bezier control polygon**
   To replace the control trigon \( \{A,B,D,E\} \) with trigons \( \{A,L,B,M\} \) and \( \{M,D,N,E\} \), each representing a portion of \( C \):
   - Insert points \( L, M, N \) at the centers of the three edges (second figure from left)
   - Move \( B \) and \( D \) to be each the average of their two neighbors (center figure)
   - Move \( M \) to be the average of its two neighbors (second figure from right)

This subdivision may be recursively applied to \( \{A,L,B,M\} \) and/or \( \{M,D,N,E\} \), as desired.

4. **Evaluating a point \( C(t) \) on a cubic Bezier curve**
   To compute \( C(t) \) perform the following sequence of operations: \( \text{slide}(E), \text{slide}(D), \text{slide}(B), \text{slide}(E), \text{slide}(D), \text{slide}(E) \), where \( \text{slide}(K) \) replaces control point \( K \) by \( (1-t)J+tK \), where \( J \) precedes \( K \) in the sequence \( \{A,B,D,E\} \). Subscripts indicate order of slides in the figure. The result of the last slide, \( E_6 \), is \( C(t) \). Note that \( C \) starts at \( A \), where it is tangent to \( AB \) and finishes at \( D \), where it is tangent to \( CD \). It is contained in the convex hull of \( \{A,B,C,D\} \).