Sampling Near-Convex Bodies

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Abstract

Given a convex body, it is known that we can sample a point uniformly from the convex body efficiently by a random walk. We would like to extend this beyond the case of convex bodies.

Computational Model

The near-convex body is given by a membership oracle. That is, given a point, the oracle replies either "Yes" if the point is present in the body or "No" if the point is not present in the body. The number of queries to the oracle is the query complexity of the algorithm. We say that an algorithm is efficient if the number of queries is polynomial in the dimensions.

Definition: Near-Convex

We consider those bodies L such that $L = K \setminus H$ where K is a convex body, while $H \subseteq K$. The bodies K and L are said to be $h-$**close** if for every line l that intersects the body K, $|l \cap H| \leq h$.

Other Definitions considered

1. If K is a convex body, then a body L is said to be ϵ-close to K if along every line l, $|l \cap (K \Delta L)| \leq \epsilon$.
2. l_1 distance - If f_K and f_L are the uniform distributions on K and L, then $\int_{\mathbb{R}^n} |f_K - f_L| \leq \epsilon$.

Remark: We expect the definition that we consider here to generalise to (2).

Idea

Proving mixing rate essentially involves proving two aspects:

- Isoperimetry: For convex bodies, we have that for any partition of $K = S_1 \cup S_2 \cup S_3$,

$$\text{vol}(S_3) \geq \frac{d(S_1, S_2)}{D} \min\{\text{vol}(S_1), \text{vol}(S_2)\}$$

  where $d(S_1, S_2)$ is the minimum distance between any two points in S_1 and S_2, and D is the diameter of the body. This is a property of the body and not the random walk. We prove that near-convex bodies satisfy an isoperimetric inequality very similar to the one above.

- Fraction of points with good local conductance: The body L could have points of very low local conductance. However, just as in the case of convex bodies, we conjecture that only a small fraction of the points have low local conductance. This is a property that depends on the random walk.

Isoperimetry

Notation: We use the prime notation to denote the subset of L while the non-prime notation denotes the subset of K, i.e., $S' \subseteq L$ denotes $S \backslash H$ where $S \subseteq K$.

Isoperimetry for near-convex bodies

Theorem 1. *Let L be a body such that there exists a convex body K which is h-close to L. Then, for any partition of $L = S'_1 \cup S'_2 \cup S'_3$,*

$$Vol(S'_3) \geq \frac{d(S_1, S_2) - h}{D} \min\{Vol(S'_1), Vol(S'_2)\}$$

where $d(S_1, S_2)$ is the shortest distance between S_1 and S_2 and D is the diameter of K.

Proof. Suppose not. Denote $C = \frac{d(S_1,S_2)-h}{D}$. By localization lemma, we get an interval $[a, b]$ and a linear function l such that

$$\int_0^1 1_{Z'_3}((1-t)a + tb)l(t)^{n-1}dt < C \int_0^1 1_{Z'_1}((1-t)a + tb)l(t)^{n-1}dt \quad \text{and}$$
$$\int_0^1 1_{Z'_3}((1-t)a + tb)l(t)^{n-1}dt < C \int_0^1 1_{Z'_2}((1-t)a + tb)l(t)^{n-1}dt.$$

where $1_{Z'_i}$ denotes the indicator function of Z'_i, $Z'_i = \{t \in [0, 1] : (1-t)a + tb \in S'_i\}$.

- The worst case of the above inequality occurs when all holes are in Z_3.
- If the holes do not exist, then the function $F : [0, 1] \to \mathbb{R}_+$, $F(t) = l(t)^{n-1}$ is unimodal.

Wlog we may assume that Z_i are intervals i.e., $Z_1 = [0, \alpha]$, $Z_3 = [\alpha, \beta]$, $Z_2 = [\beta, 1]$ and $\int_{Z_1} l(t)^{n-1} < \int_{Z_2} l(t)^{n-1}$. Then, we know that

$$\begin{aligned}\int_{Z_3} F(t)dt &\geq (\beta - \alpha - h)F(\alpha) \\ &\geq (\beta - \alpha - h)/\alpha \int_{Z_1} F(t)dt \\ &> (\beta - \alpha - h) \int_{Z_1} F(t)dt \\ &= (d(Z_1, Z_2) - h) \int_{Z_1} F(t)dt\end{aligned}$$

Suppose Z_i are not intervals. Let $Z_3 = \cup_i(\alpha_i, \beta_i)$. Then, by previous case,

$$\int_{\alpha_i}^{\beta_i} F(t)dt \geq d(Z_1, Z_2) - hmin\{\int_0^{\alpha_i} F(t)dt, \int_{\beta_i}^1 F(t)dt\}$$

Summing this over i, we get

$$\int_{Z_3} F(t)dt \geq d(S_1, S_2) - h \sum_{i=1}^{k} min\{\int_0^{\alpha_i} F(t)dt, \int_{\beta_i}^1 F(t)dt\}$$

Since every point of Z_1 and every point of Z_2 are separated by at least one of the intervals (α_i, β_i),

$$\sum_{i=1}^{k} min\{\int_0^{\alpha_i} F(t)dt, \int_{\beta_i}^{1} F(t)dt\} \geq min\{\int_{Z_1} F(t)dt, \int_{Z_2} F(t)dt\}$$

Thus,

$$Vol(S_3') \geq \frac{d(S_1, S_2) - h}{D} \min\{Vol(S_1'), Vol(S_2')\}$$

□

Local Conductance

Local Conductance, $l(x)$ at a point $x \in L$ is the probability of moving out of x to some other point in L according to the ball walk.

$$l(x) := \frac{\text{vol}\,(L \cap x + r\mathbb{B})}{\text{vol}\,(r\mathbb{B}_0)}$$

For a radius r of the ball walk on a body L which is h-close to a convex body K, define $L(r,p) = \{u \in L : l(u) \geq p\}$. Suppose L contains the unit ball in $\mathbb{R}^n$. We know that for a convex body K, the following are true:

- $K(r,p)$ is convex.
- $\text{vol}\,(K(r,p)) \geq (1 - \frac{r\sqrt{n}}{2(1-p)})\,\text{vol}\,(K)$.

For a body L which is h-close to K, we conjecture the following:

Conjecture 1. *For $L(r,p)$ as defined above,*

- *$L(r,p)$ is $2h$-close to K.*
- $\text{vol}\,(L(r,p)) \geq (1 - p_1(h)\frac{r\sqrt{n}}{2(1-p)})\,\text{vol}\,(L)$

for some simple function $p_1(\cdot)$ of h.