CS 1050: Constructing Proofs

Solutions to Problem Set 1

Problem 1

Direct Proofs (10 points)

1. Prove that for every real number x, $x^2 - 2x + 2 > 0$.

   Proof. $x^2 - 2x + 2 = (x-1)^2 + 1$
   $(x-1)^2 \geq 0$
   $\Rightarrow (x-1)^2 + 1 > 0$ $\square$

2. Suppose a and b are integers such that $a - b$ is divisible by 7. Prove that $a^2 - b^2$ is also divisible by 7. Prove that $a^3 - b^3$ is also divisible by 7.

   Proof. $a^2 - b^2 = (a-b)(a+b)$
   Since $(a-b)$ is divisible by 7, therefore $(a^2 - b^2)$ is also divisible by 7.
   $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
   Since $(a-b)$ is divisible by 7, therefore $(a^3 - b^3)$ is also divisible by 7.
   $\square$

Problem 2

Examples/Counter-examples (20 points)

Prove or disprove each of the following propositions by giving an example or a counter-example.

1. There is a natural number p such that $p, p+2, p+6$ and $p+8$ are all primes.

   True. Let $p = 5$. Then, $p = 5, p+2 = 7, p+6 = 11$ and $p+8 = 13$ are all primes. $\square$

2. If x and y are positive irrational numbers, then $x \cdot y$ is also an irrational number.

   False. Let $x = \sqrt{2}, y = \sqrt{2}$. Then $x \cdot y = \sqrt{2} \cdot \sqrt{2} = 2$ is rational. $\square$

3. If x and y are positive rational numbers, then x^y is also a rational number.

   False. Let $x = 2, y = \frac{1}{2}$. Then $x^y = 2^{\frac{1}{2}} = \sqrt{2}$ is irrational. $\square$

4. If A and B are two finite sets, then $|A \cup B| = |A| + |B|$. Here $|A|$ denotes the number of elements in the set A.

   False. Let $A = \{1,2,3,4\}, B = \{3,4,5\}$. Then, $A \cup B = \{1,2,3,4,5\}$. In this case, $RHS = |A| + |B| = 4 + 3 = 7$ and $LHS = |A \cup B| = 5$. $\square$

Problem 3

Proofs by Contrapositive (10 points)

1. Let x and y be real numbers. Prove that if $x+y$ is irrational, then either x or y is irrational.

   Proof. We prove the contrapositive, i.e. we prove that if x and y both are rational then $x+y$ is also rational.

   Let x and y be both rational. Then we can write:

   $$\begin{aligned} x &= \frac{a}{b} \quad \text{where} \quad b \neq 0 \\ y &= \frac{c}{d} \quad \text{where} \quad d \neq 0 \\ \Rightarrow x+y &= \frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd} \quad \text{where} \quad bd \neq 0 \end{aligned}$$

   So, $x+y$ is rational. ☐

2. Let a and b be integers. Prove that if $a \cdot b$ is odd, then a and b both are odd.

Proof. We prove the contrapositive, i.e. if either a or b is even, then $a \cdot b$ is even.

Let a be even (the case when b is even is similar). Then, $a = 2 \cdot x$ for some integer x. Now, $a \cdot b = (2 \cdot x) \cdot b = 2 \cdot (b \cdot x)$. Therefore, $a \cdot b$ is even.

☐

Problem 4

Proofs by Contradiction (10 points)

1. Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Let y be their average, i.e. $y = \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)$. Prove that $y \leq x_i$ for some $i \in \{1, 2, \ldots, n\}$.

   Proof. Suppose on the contrary that $y > x_i \forall i \quad \in \; \{1, 2, \ldots, n\}$.

   $$\begin{aligned} \therefore \quad n \cdot y &> \left(\sum_{i=1}^{n} x_i\right) \\ \therefore \quad y &> \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right) \quad \text{CONTRADICTION as} \quad y = \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right) \end{aligned}$$

   Therefore, $y \leq x_i$ for some $i \in \{1, 2, \ldots, n\}$. ☐

2. Prove that for any integer n, at least one of the three integers $n, 2n+1, 4n+3$ is not divisible by 7.

Proof. Let all of $n, 2n+1, 4n+3$ be divisible by 7.
$\Rightarrow n = 7a$ for some integer a.
$2n+1 = 7b$ for some integer b.
$4n+3 = 7c$ for some integer c.
Therefore, adding the three equations we get $7n+4 = 7(a+b+c)$ which implies $4 = 7(a+b+c-n)$ which is a contradiction since 4 is not divisible by 7.

□

Problem 5

Proofs by Cases (10 points)

1. For any real numbers x and y, prove that $|x+y| \leq |x| + |y|$. Here $|x|$ denotes the absolute value of x which equals x if $x \geq 0$ and equals $-x$ if $x < 0$. (*Hint: There are four cases depending on whether x and y are greater-or-eqaul/less compared to zero*).

   Proof. Case 1 : $x \geq 0, y \geq 0$
   $|x| = x$ as $x \geq 0$
   $|y| = y$ as $y \geq 0$
   $|x+y| = x+y$ as $x \geq 0, y \geq 0 \Rightarrow x+y \geq 0$.
   Therefore, $|x+y| = |x| + |y|$.

   Case 2 : $x < 0, y < 0$
   $|x| = -x$ as $x < 0$
   $|y| = -y$ as $y < 0$
   $|x+y| = -(x+y)$ as $x < 0, y < 0 \Rightarrow x+y < 0$.
   Therefore, $|x+y| = |x| + |y|$.

   Case 3 : $x \geq 0, y < 0$
   $|x| = x$ as $x \geq 0$
   $|y| = -y$ as $y < 0$
   $y \leq x+y < x$ as $x \geq 0, y < 0$
   If $x+y \geq 0$ then $|x+y| = x+y < x = |x|$. Therefore, $|x+y| < |x| + |y|$ (just adding a positive number to RHS).
   If $x+y < 0$ then $|x+y| = -(x+y) \leq -y = |y|$. Therefore, $|x+y| \leq |x| + |y|$ (just adding a positive number to RHS).

   Case 4 : $x < 0, y \geq 0$
   This is similar to Case 3 and can be proved similarly.

   □

2. If n is a natural number (written in usual decimal notation), prove that the last digit of n^4 is either $0, 1, 5$ or 6.

Proof. If n is a natural number, then it can be written as $10 \cdot a + i$ where $i \in \{0, 1, 2, \dots, 9\}$. Therefore, $n^4 = (10 \cdot a + i)^4 = 10 \cdot k + i^4$ where k is rest of the expansion. So last digit of n^4 is same as that of i^4.

Case 0 : $i = 0 \Rightarrow i^4 = 0 \Rightarrow$ last digit of $n^4 = 0$
Case 1 : $i = 1 \Rightarrow i^4 = 1 \Rightarrow$ last digit of $n^4 = 1$
Case 2 : $i = 2 \Rightarrow i^4 = 16 \Rightarrow$ last digit of $n^4 = 6$
Case 3 : $i = 3 \Rightarrow i^4 = 81 \Rightarrow$ last digit of $n^4 = 1$
Case 4 : $i = 4 \Rightarrow i^4 = 256 \Rightarrow$ last digit of $n^4 = 6$
Case 5 : $i = 5 \Rightarrow i^4 = 625 \Rightarrow$ last digit of $n^4 = 5$
Case 6 : $i = 6 \Rightarrow i^4 = 1296 \Rightarrow$ last digit of $n^4 = 6$
Case 7 : $i = 7 \Rightarrow i^4 = 2401 \Rightarrow$ last digit of $n^4 = 1$
Case 8 : $i = 8 \Rightarrow i^4 = 4096 \Rightarrow$ last digit of $n^4 = 6$
Case 9 : $i = 9 \Rightarrow i^4 = 6561 \Rightarrow$ last digit of $n^4 = 1$

□

Problem 6

Puzzle (10 points)

There are seven jars with seven coins in each jar. In one of the jars, all coins are fake. All coins in all the remaining six jars are true coins. A fake coin looks exactly like a true coin but weighs less. A fake coin weighs 9 grams whereas a true coin weighs 10 grams. You are given a weighing scale. How will you find out which jar has fake coins with only one weighing?

Select 1 coin from first jar, 2 coins from second jar, 3 coins from third jar, 4 coins from fourth jar, 5 coins from fifth jar, 6 coins from sixth jar, and all 7 coins from seventh jar.

Total number of coins selected is $1 + 2 + \ldots + 7 = 28$. If all coins were true then the total weight of selected coins would have been 280 grams. However coins in the k^{th} jar are fake (k is unknown). Since we selected k coins from k^{th} jar, and each fake coin weighs 1 gram less, the total weight of selected coins will be $280 - k$ grams. Thus, we can weigh the selected coins and if their total weight is $280 - i$ grams for some i, then declare that the i^{th} jar has fake coins.