CS 1050A: Constructing Proofs

Problem Set 2

Due Friday, Sept 15th, after the class

All problems are worth 10 points.

Problem 1

Let A, B, C be sets (subsets of some fixed universe). Let $A \Delta B$ denote the *symmetric difference* of the two sets, defined to be the set of all elements that occur either in A or in B but not in both.

1. Draw the Venn diagram representing $A \Delta B$.
2. Prove that $(A \Delta B) \Delta C = A \Delta (B \Delta C)$. You can prove using a Venn diagram or otherwise.

Problem 2

Let A, B, C be finite sets (subsets of some fixed universe). Recall that $|A|$ denotes the number of elements in the set A. Prove that

1. $|A \cup B| = |A| + |B| - |A \cap B|$.
2. $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$.

Problem 3

For each of the following functions, decide whether they are one-to-one and/or onto. Why?

1. $f : \mathbb{Z} \mapsto \mathbb{Z}$ defined as $f(x) = x^3$.
2. $f : \mathbb{R} \mapsto \mathbb{R}$ defined as $f(x) = x^3$.
3. $f : \mathbb{R} \mapsto \mathbb{Z}$ defined as $f(x) = \lfloor x \rfloor$. Here $\lfloor x \rfloor$ denotes the *integral part* of x defined to be the largest integer less than or equal to x.
4. $f : \mathbb{Z} \mapsto \mathbb{Z}$ defined as $f(x) = x^2 - 3x$.

Problem 4

Let $x_1, x_2, \ldots, x_n$ be n positive real numbers between 0 and 1. Let their average be μ, i.e. $\mu = \frac{x_1+x_2+\ldots+x_n}{n}$. (*Hint: Prove by contradiction*).

1. Prove that at most $\lfloor \frac{n}{2} \rfloor$ of these numbers are greater than 2μ.
2. Prove that at least $\lfloor \frac{\mu n}{2} \rfloor$ of these numbers are greater than or equal to $\frac{\mu}{2}$.

Problem 5

Let $f : X \mapsto Y$ and $g : Z \mapsto X$ be two functions.

1. If f and g are both onto, then show that $f \circ g$ is also onto.

2. Suppose that $f \circ g$ is onto. Is f necessarily onto? Is g necessarily onto?

Problem 6

Puzzle: A hiker started climbing a mountain at 8:00 am and reached the top at 6:00 pm. She spent the night there and started climbing down (on the same trail) the next morning at 8:00 am. She reached the bottom at 6:00 pm. On both days, she hiked at uneven/varying speed and rested several times. Prove that at *some* time on both the days (like 2:13 pm on both days) she was at the same exact spot on the hiking trail. (*Hint: Superimpose the scenarios on the two days).*

Problem 7

Feedback

1. How did you find this homework? Too easy? Too difficult? Just right?

2. How do you like the teaching so far? Too slow? Too fast? Just right? Any suggestions?