CS1050 HW 2 Solutions

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Problem 2

1. Let the universe of elements be $\mathcal{U}$. We know that $A \cup B$ contains all elements of $\mathcal{U}$ which belong to atleast one of A or B. Similarly, we know that $A \cap B$ contains all elements of $\mathcal{U}$ which belong to both A and B. Say, S_A is the set of elements that belong to A but not to B and S_B is the set of elements that belong to B but not to A. Clearly, S_A is the set formed after removing the elements of the set $A \cap B$ from A, and since $A \cap B \subseteq A$, $|S_A| = |A| - |A \cap B|$. Similarly, $|S_B| = |B| - |A \cap B|$. Now, $A \cup B = S_A \cup S_B \cup (A \cap B)$, and since $S_A,\ S_B$ and $A \cap B$ are disjoint,

$$\begin{aligned} |A \cup B| &= |S_A| + |S_B| + |A \cap B| \\ &= |A| - |A \cap B| + |B| - |A \cap B| + |A \cap B| \\ &= |A| + |B| - |A \cap B| \end{aligned}$$

2. We can write $A \cup B \cup C = A \cup (B \cup C)$, and using the formula in part 1, we get,

$$\begin{aligned} |A \cup B \cup C| &= |A \cup (B \cup C)| \\ &= |A| + |B \cup C| - |A \cap (B \cup C)| \\ &= |A| + |B \cup C| - |(A \cap B) \cup (A \cap C)| \end{aligned}$$

Now, applying part 1 again we have that $|B \cup C| = |B| + |C| - |B \cap C|$, and $|(A \cap B) \cup (A \cap C)| = |A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)| = |A \cap B| + |A \cap C| - |A \cap B \cap C|$. Substituting these formulas we have,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

which is what we want.

Problem 3

1. $f : \mathbf{Z} \mapsto \mathbf{Z}$, $f(x) = x^3$ is one-to-one, since if $a^3 = b^3$, then $\left(\frac{a}{b}\right)^3 = 1$, where $a, b \in \mathbf{Z}$ and so $\frac{a}{b} = 1$ and so $a = b$. But f is not onto, since suppose $a^3 = 2$, then $a = 2^{\frac{1}{3}}$, which is not an integer.

2. $f : \mathbf{R} \mapsto \mathbf{R}$, $f(x) = x^3$, is one to one, since if $a^3 = b^3$ then $\left(\frac{a}{b}\right)^3 = 1$, where $a, b \in \mathbf{Z}$ and so $\frac{a}{b} = 1$ and so $a = b$. Also, f is onto, since if $b \in \mathbf{R}$ then $b^{\frac{1}{3}} \in \mathbf{R}$.
3. $f : \mathbf{R} \mapsto \mathbf{Z}$, $f(x) = \lfloor x \rfloor$, is not one to one since $f(1.1) = f(1.2) = 1$. But it is onto since $f(a) = a$ for all $a \in \mathbf{Z}$.
4. $f : \mathbf{Z} \mapsto \mathbf{Z}$, $f(x) = x^2 - 3x$, is not one to one since $f(0) = f(3) = 0$. Also, f is not onto since suppose $x^2 - 3x = -1$, then the roots of this equation are $(3 + \sqrt{5})/2$ and $(3 - \sqrt{5})/2$, both of which are not integers. So, there is no integer a such that $f(a) = -1$.

Problem 4

1. Suppose for a contradiction that there are $\lfloor \frac{n}{2} \rfloor + 1$ of the numbers with value greater than 2μ. Therefore the total sum of these numbers is atleast $\left(\lfloor \frac{n}{2} \rfloor + 1\right) 2\mu > \left(\frac{n}{2}\right) 2\mu = n\mu = x_1 + x_2 + \cdots + x_n$ which is a contradiction. Therefore, atmost $\lfloor \frac{n}{2} \rfloor$ of these numbers have value greater than 2μ.
2. Suppose for a contradiction that atmost t $(t \leq \lfloor \frac{\mu n}{2} \rfloor - 1)$ numbers are greater or equal to $\frac{\mu}{2}$. The sum of the rest of the numbers is atmost $(n - t)\frac{\mu}{2}$. Since the numbers are between 0 and 1, the total sum is atmost $(n - t)\frac{\mu}{2} + t = \frac{\mu n}{2} + t(1 - \frac{\mu}{2})$. Now since $\mu \leq 1$, the sum is atmost $\frac{\mu n}{2} + (\lfloor \frac{\mu n}{2} \rfloor - 1)(1 - \frac{\mu}{2}) < \frac{n\mu}{2} + \frac{n\mu}{2} = \mu n$, which is a contradiction. Therefore, atleast $\lfloor \frac{\mu n}{2} \rfloor$ of the numbers are greater than or equal to $\frac{\mu}{2}$.

Problem 4

1. Let $y \in Y$. Since f is onto there is a $x \in X$ such that $f(x) = y$, and since g is onto, there is a $z \in Z$ such that $g(z) = x$. Therefore, $f \circ g(z) = f(g(z)) = f(x) = y$, so $f \circ g$ is onto.
2. Suppose $f \circ g$ is onto. Then, let $y \in Y$ be an element. Since $f \circ g$ is onto, there is a $z \in Z$ such that $f \circ g(z) = f(g(z)) = y$. Therefore, setting $x = g(z)$, we get a $x \in X$ such that $f(x) = y$. Therefore, f is onto. However, consider the example: let $X = Y = Z = \mathbf{Z}$ and let $f(x) = \lfloor \frac{x}{2} \rfloor$ and $g(z) = 2z$. Therefore, $f \circ g(z) = f(g(z)) = f(2z) = z$ which is onto, but $g(z) = 2z$ is not onto since not every integer is even.

Problem 6

If we superimpose the hiker's climb and the descent along the same trail, each starting at 8:00 am and ending at 6:00 pm, there must be a point of time between 8:00 am and 6:00 pm, when the climb and the descent crosses (along the same trail). At this point of time the climber was in the same position in her climb as well as descent.