CS 1050A: Constructing Proofs

Problem Set 3

Due on Oct 11, 2006

Problem 1 (10 points)

Prove that $((p \to q) \wedge (q \to r)) \to (p \to r)$ is a tautology (using truth table or otherwise).

Problem 2 (10 points)

Determine the truth value of each of these quantified statements if the domain of each variable consists of all real numbers. Provide necessary justification.

1. $\forall x \exists y \;\; (x = y^2)$
2. $\exists x \exists y \;\; (x + 2y = 2 \;\wedge\; 2x + 3y = 5)$

Problem 3 (20 points)

Prove by induction that for every integer $n \geq 1$,

1. $1^2 + 2^2 + 3^2 + \ldots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}$
2. $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$

Problem 4 (10 points)

Let $h > 0$ be a (fixed) real number. Prove by induction that for every integer $n \geq 1$,

$$(1+h)^n \geq \; 1 + hn$$

Problem 5 (10 points)

Suppose there are n lines in plane ($n \geq 1$) such that no two lines are parallel and no three lines intersect at a common point. Prove that these lines divide the plane into $\frac{n(n+1)}{2} + 1$ regions. (*Hint: Use induction*).

Problem 6 (10 points)

Define a function $f : \mathbb{N} \mapsto \mathbb{Z}$ recursively as follows:

$$f(0) = 1, \quad f(1) = 4, \qquad \text{and} \quad \forall\ n \geq 2,\ f(n) = 4f(n-1) - 4f(n-2)$$

Prove by induction that $\forall\ n \geq 0,\ f(n) = 2^n(n+1)$.

Problem 7 (10 points)

Find a flaw in the following "proof" that all horses have the same color.

Let $P(n)$ be the proposition that all horses in a set of n horses have the same color. We will prove by induction that $P(n)$ is true for every integer $n \geq 1$.
(Base case): Clearly $P(1)$ is true since it is just a single horse.
(Inductive step): Assume that $P(k)$ is true for some $k \geq 1$, i.e. for any set of k horses, they all have the same color. We will prove that $P(k+1)$ is true.

Well, given any set of $k+1$ horses, if you exclude the *last* horse, you get a set of k horses. By the inductive hypothesis, these k horses all have the same color. Similarly, by excluding the *first* horse, you can conclude that the last k horses have the same color. Since the sets of first k horses and the last k horses overlap, all $k+1$ horses have the same color. Hence proved.