CS 1050A: Constructing Proofs

Problem Set 5

Due on Nov 29, 2006

Problem 1

Let $f, g, h : \mathbb{N} \mapsto \mathbb{N}$ be functions such that $f(n)$ is $O(g(n))$ and $h(n)$ is $O(g(n))$. Prove that

1. $f(n) + h(n)$ is $O(g(n))$.
2. $f(n) \cdot h(n)$ is $O(g(n)^2)$.

Problem 2

Let a, b, c, d be positive integers such that $a|c$ and $b|d$.

1. Prove that $ab|cd$.
2. Prove that $\gcd(a, b)|\gcd(c, d)$.

Problem 3

Compute $\gcd(150, 42)$ by each of these three methods:

1. Writing down the sets of divisors of 150 and 42.
2. Finding the prime factorizations of 150 and 42.
3. Using Euclid's algorithm. Also, find integers s and t such that $\gcd(150, 42) = s \cdot 150 + t \cdot 42$.

Problem 4

Let $a, b, c, n \in \mathbb{Z}$ and $n > 0$. Prove that if

$$a \equiv b \pmod{n}, \quad \text{and} \quad b \equiv c \pmod{n}$$

then $a \equiv c \pmod{n}$.

Problem 5

Let a and b be positive integers. Let $\text{lcm}(a, b)$, the *least common multiple* of a and b, denote the least positive integer m such that m is divisible by both a and b.

1. What is $\text{lcm}(6, 9)$? What is $\text{lcm}(60, 18)$?

2. Suppose a and b have the following prime factorizations:

$$a = \prod_{i=1}^{k} p_i^{t_i}, \qquad b = \prod_{i=1}^{k} p_i^{s_i}$$

   where $p_1, p_2, \ldots, p_k$ are distinct primes, and $t_i, s_i \geq 0$ are integers. What is the prime factorization of $\text{lcm}(a, b)$?

3. Prove that for any positive integers a and b,

$$a \cdot b = \gcd(a, b) \cdot \text{lcm}(a, b).$$