CS 1050A: Constructing Proofs

Solutions to Quiz 1

Problem 1 (20 points)

For this problem, you can assume that $\sqrt{2}$ is irrational (which we proved in class).

Let x and y be two rational numbers such that $x < y$. Let $z = \frac{\sqrt{2}\ x\ +\ (2-\sqrt{2})\ y}{2}$.

1. Prove that $x < z$.

   Proof. (By contradiction)
   Assume on the contrary that $x \geq z$
   $\Rightarrow x \geq \frac{\sqrt{2}\ x\ +\ (2-\sqrt{2})\ y}{2}$
   $\Rightarrow (2-\sqrt{2})\ x \geq (2-\sqrt{2})\ y$
   $\Rightarrow x \geq y$ (Contradiction since it is given that $x < y$)
   Therefore, $x < z$. □

2. Prove that $z < y$.

   Proof. (By contradiction)
   Assume on the contrary that $z \geq y$
   $\Rightarrow y \leq \frac{\sqrt{2}\ x\ +\ (2-\sqrt{2})\ y}{2}$
   $\Rightarrow (2-\sqrt{2})\ y \leq (2-\sqrt{2})\ x$
   $\Rightarrow y \leq x$ (Contradiction since it is given that $x < y$)
   Therefore, $z < y$. □

3. Prove by contradiction that z is irrational.

   Proof. (By Contradiction) Let z be a rational number. Therefore, $z = \frac{p}{q}$, where p, q are integers and $q \neq 0$.
   $\Rightarrow \frac{p}{q} = \frac{\sqrt{2}\ x\ +\ (2-\sqrt{2})\ y}{2}$
   $\Rightarrow 2p = \sqrt{2}xq + 2yq - \sqrt{2}yq$
   $\Rightarrow 2(p - yq) = \sqrt{2}(xq - yq)$
   $\Rightarrow \sqrt{2} = \frac{q(x-y)}{p-yq}$ - a rational number (Contradiction as $\sqrt{2}$ is irrational)
   Therefoe, z is irrational.

   □

4. Prove that there exists a rational number w such that $x < w < y$.

   Proof. Let $w = \frac{x+y}{2}$. w is rational as it is average of two rational numbers.
   Claim : $x < w$
   $x < y$
   $\Rightarrow 2x < x + y$

$\Rightarrow x < \frac{x+y}{2} = w$

Claim : $w < y$
$x < y$
$\Rightarrow x + y < 2y$
$\Rightarrow w = \frac{x+y}{2} < y$

Therefore, there exists a rational number w such that $x < w < y$.

$\square$

Problem 2 (20 points)

Let $f : \{A, B, C, D\} \mapsto \{1, 2, 3\}$ be a function defined as

$$f(A) = 2,\ f(B) = 3,\ f(C) = 3,\ f(D) = 2$$

Let $g : \{1, 2, 3\} \mapsto \{T, F\}$ be a function defined as

$$g(1) = T,\ g(2) = F,\ g(3) = F$$

Answer the following questions by encircling your answer (no explanation necessary).

1. Is f onto?

   NO, there is no $x \in \{A, B, C, D\}$ such that $f(x) = 1$.

2. Is g onto?

   YES, $g(1) = T,\ g(2) = F$.

3. Is $g \circ f$ onto?

   NO, there is no $x \in \{A, B, C, D\}$ such that $(g \circ f)(x) = T$.

4. Is $g \circ f$ one-to-one?

   NO, as $(g \circ f)(A) = (g \circ f)(B) = (g \circ f)(C) = (g \circ f)(D) = F$.

Problem 3 (10 points)

Let p and q be propositional variables. Prove that the compound proposition $(p \wedge (p \rightarrow q)) \rightarrow q$ is a tautology (i.e. it evaluates to TRUE for all possible truth values of the variables). You can prove using the truth table or otherwise.

Truth Table for $(p \wedge (p \rightarrow q)) \rightarrow q$ is as follows (here 0 represents FALSE and 1 represents TRUE):

p	q	$(p \rightarrow q)$	$p \wedge (p \rightarrow q)$	$(p \wedge (p \rightarrow q)) \rightarrow q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1