CS 1050A: Constructing Proofs

Solutions to Quiz 2

Problem 1 (15 points)

Prove by induction that for every $n \geq 1$,

$$1 + 3 + 3^2 + 3^3 + \ldots + 3^{n-1} + 3^n = \frac{3^{n+1} - 1}{2}$$

Proof. Base Case :$n = 1$
$LHS = 1 + 3^1 = 4$
$RHS = \frac{3^{1+1}-1}{2} = \frac{9-1}{2} = 4$
Since, $LHS = RHS$, the proposition is true for $n = 1$.

Inductive Hypothesis : Assume that the proposition is true for $n = k$ i.e.
$1 + 3 + 3^2 + 3^3 + \ldots + 3^{k-1} + 3^k = \frac{3^{k+1}-1}{2}$.

Inductive Step : $n = k + 1$
$1 + 3 + 3^2 + 3^3 + \ldots + 3^k + 3^{k+1} = \frac{3^{k+1}-1}{2} + 3^{k+1}$
$= \frac{3^{k+1}-1+2\cdot 3^{k+1}}{2}$
$= \frac{3^{k+2}-1}{2}$

Therefore, the proposition is true for all integers $n \geq 1$.

□

Problem 2 (10 + 5 + 10 + 5 + 5 points)

Let $f : \mathbb{Z}^+ \mapsto \mathbb{Z}^+$ be a function defined as follows:

$$f(1) = 1$$

$$\forall\ n \geq 2, \qquad f(n) = f(n-1) + 2n - 1$$

1. Compute the values $f(2), f(3), f(4), f(5), f(6)$.

   $f(2) = f(1) + 2 \cdot 2 - 1 = 1 + 4 - 1 = 4$
   $f(3) = f(2) + 2 \cdot 3 - 1 = 4 + 6 - 1 = 9$
   $f(4) = f(3) + 2 \cdot 4 - 1 = 9 + 8 - 1 = 16$
   $f(5) = f(4) + 2 \cdot 5 - 1 = 16 + 10 - 1 = 25$

2. Guess the "formula" for $f(n)$.

   $f(n) = n^2$

3. Prove by induction that the formula is correct for every $n \geq 1$.

Proof. *Base Case* : $n = 1$
$LHS = f(1) = 1$ (Given)
$RHS = 1^2 = 1$
Since, $LHS = RHS$, the proposition is true for $n = 1$.

Inductive Hypothesis : Assume that the proposition is true for $n = k$ i.e. $f(k) = k^2$.

Inductive Step : $n = k + 1$
$f(k + 1) = f(k) + 2(k + 1) - 1 = k^2 + 2k + 1$
$= (k + 1)^2$

Therefore, $f(n) = n^2 \forall n \geq 1$.

□

4. Prove by induction that for every $n \geq 1$,

$$f(n) = \sum_{i=1}^{n}(2i - 1),$$

i.e. $f(n)$ equals the sum of first n odd positive integers.

Proof. *Base Case* : $n = 1$
$LHS = f(1) = 1$ (Given)
$RHS = \sum_{i=1}^{1}(2i - 1) = 2 - 1 = 1$
Since, $LHS = RHS$, the proposition is true for $n = 1$.

Inductive Hypothesis : Assume that the proposition is true for $n = k$ i.e. $f(k) = \sum_{i=1}^{k}(2i - 1)$.

Inductive Step : $n = k + 1$
$f(k + 1) = f(k) + 2(k + 1) - 1 = \sum_{i=1}^{k}(2i - 1) + (2(k + 1) - 1)$
$= \sum_{i=1}^{k+1}(2i - 1)$

Therefore, $f(n) = \sum_{i=1}^{n}(2i - 1), \forall n \geq 1$. □

5. What is the value of the following sum? Just write the answer.

$$1 + 3 + 5 + 7 + 9 + \ldots + 995 + 997 + 999$$

$1 + 3 + 5 + 7 + 9 + \ldots + 995 + 997 + 999 = \sum_{i=1}^{500}(2i - 1)$
$= f(500)$
$= 500^2$
$= 250,000$