

Homework II

CS6520

Computational Complexity

February 12, 2007

Due on Thu, Feb 22nd, at 6 pm. Collaboration is allowed; please mention your collaborators.

1. Show that $\Sigma_k = \text{NP}^{\text{CIRCUIT-SAT}_{k-1}}$.

We define $L \in \Sigma_k$ if and only if there is a deterministic polynomial time verifier V such that

$$x \in L \iff \exists y_1 \forall y_2 \cdots Q_k y_k \ V(x, y_1, \dots, y_k) = 1$$

where length of y_1, y_2, \dots, y_k is bounded by a polynomial in $|x|$.

2.
 - Show that $\text{P}^{\text{PSPACE}} = \text{NP}^{\text{PSPACE}} = \text{PSPACE}$
 - Show that if $\text{PH} = \text{PSPACE}$, then PH collapses to some finite level.
 - Can PH have a complete problem (complete under polynomial time reductions) ?
3. (**DP-completeness**) This problem studies the class DP (D stands for difference). A language $L \in \text{DP}$ if and only if there are languages $B \in \text{NP}$ and $C \in \text{coNP}$ so that $L = B \cap C$.
 - The problem SAT-UNSAT is defined as follows: Given Boolean formulae ϕ, ψ , decide if ϕ is satisfiable And ψ is unsatisfiable. Show that this problem is DP -complete (under polynomial time reductions).
 - A graph G is in HC-CRITICAL if G is not Hamiltonian but adding any edge to G will make it Hamiltonian. Show that HC-CRITICAL is in DP .

4.
 - Show that $\text{NP}^{\text{BPP}} \subseteq \text{BPP}^{\text{NP}}$ (*Hint : First show that a language in NP^{BPP} is accepted by a polytime NTM that makes a single query to a BPP oracle and that too at the end*).
 - Show that if $\text{NP} \subseteq \text{BPP}$, then PH collapses to BPP.

5. (**NEXP-completeness**) Define $\text{NEXP} = \cup_{k=1,2,\dots} \text{NTIME}(2^{n^k})$.

Show that the following problem is NEXP-complete : Given $\langle M, x, n \rangle$, consisting of description of a NTM M , input x and an integer n in binary, does M have an accepting computation on x in n steps.

6. A circuit C is called an *implicit representation* of another circuit C^* if C takes as input a binary integer i such that $n + 1 \leq i \leq N$, and outputs a triple (TYPE, j, k) where
 - Input to C^* is an n -bit string $x_1x_2 \dots x_n$.
 - $\text{TYPE} \in \{\text{AND}, \text{OR}, \text{NOT}\}$ indicates the type of i^{th} gate in circuit C^* .
 - $1 \leq j, k \leq N$.
 - The input of the i^{th} gate in C^* is the output of the j^{th} and k^{th} gates of C^* (if $\text{TYPE} = \text{NOT}$, then k is ignored. If $1 \leq j, k \leq n$, then the j^{th} or k^{th} gate is taken to be an input bit x_i).
 - The N^{th} gate in C^* is its output gate.

Note that we could have $N = 2^n$, the circuit C could be of size $\text{poly}(\log N) = \text{poly}(n)$ and still implicitly represent a circuit C^* of size N (in short, a circuit can implicitly represent another circuit of size exponential in its own size).

Let IMPLICIT CIRCUIT-SAT be the following problem : Given a circuit C that is an implicit representation of circuit C^* , decide if C^* is satisfiable. Show that this problem is NEXP-complete (*Hint : Use the regular structure of the circuit produced in Cook's reduction*).

7.
 - Show that $\text{NP}^{\text{NP} \cap \text{coNP}} = \text{NP}$.
 - Generalize this to $\text{NP}^{\Sigma_k \cap \Pi_k} = \Sigma_k$.
8. The problem Graph Consistency (GC) asks, for two given sets A and B of graphs, whether there exists a graph G such that every graph $g \in A$ is isomorphic to a (not necessarily induced) subgraph of G but each graph $h \in B$ is not isomorphic to any subgraph of G . Show that GC is in Σ_2 . (Optional : show that it is Σ_2 -complete).

9. Show that if $\Sigma_k = \Pi_k$ for some k , then $\text{PH} = \Sigma_k$.