Continuous-Time Continuous-Valued Random Processes

Guy Lebanon

January 6, 2006

We focus on continuous-time continuous-valued RPs with the prototypical RP being the Gaussian process. We start by recalling that a linear transformation of a vector RV with multivariate Normal distribution is multivariate normal. For example, a normal random vector $\vec{X}$ with 0 means linearly transformed by the matrix $T$ (i.e. $\vec{Y} = T\vec{X}$) has pdf

$$f_{\vec{Y}}(\vec{y}) = \frac{1}{|\det T|} \frac{1}{(2\pi)^{n/2}\sqrt{\det \Sigma}} e^{-\frac{1}{2}(T^{-1}\vec{y})^\top T^{-1}\Sigma^{-1}(T^{-1}\vec{y})}$$

$$= \frac{1}{(2\pi)^{n/2}\sqrt{\det T^\top \Sigma T}} e^{-\frac{1}{2}\vec{y}^\top T^{-1}\Sigma^{-1}T^{-1}\vec{y}}$$

where in the last equality we used the fact that $\sqrt{\det T^\top \Sigma T} = \sqrt{\det T} \det \Sigma \det T = |\det T| \sqrt{\det \Sigma}$. It follows then that $T\vec{X}$ is distributed multivariate normal with zero mean and $T^\top \Sigma T$ covariance matrix. A similar (but messier) proof holds for a linearly transformed random vector $T\vec{X}$ whose mean vector is not zero.

**Definition 1.** A random process $\mathcal{X} = \{X_t : t \in A \subset \mathbb{R}\}$ is a Gaussian process if all of its finite dimensional marginals are multivariate normal random vectors:

$$\forall k > 0, \forall t_1, \ldots, t_k \in A, \quad f_{X_{t_1}, \ldots, X_{t_k}}(\vec{a}) = \frac{1}{(2\pi)^{k/2}\sqrt{\det \Sigma}} e^{(\vec{a} - \vec{\mu})^\top \Sigma^{-1}(\vec{a} - \vec{\mu})}.$$

The mean vector and covariance matrix above may change depending on the selection of $k, t_1, \ldots, t_k$. However, they must be consistent in the sense described previously in class.

The mean function $m_X(t)$ and the auto-covariance function $C_X(t, s)$ define the mean of all the marginals RVs $X_{t_1}, \ldots, X_{t_k}$ as well as the covariance matrix and as a result they define the pdf for all the finite dimensional marginals of the Gaussian process. As a result, for a Gaussian process, the mean and auto-covariance function define it uniquely by Kolmogorov’s theorem.

An interesting special case of the Gaussian process is the Wiener process.

**Definition 2.** The Wiener process $\mathcal{Z} = \{Z_t : t \geq 0\}$ is a Gaussian process with $Z_0 = 0$ identically, $m_Z(t) = 0$ and $C_Z(t, s) = \alpha \min(t, s)$ for some $\alpha > 0$.

We motivate the Wiener process by deriving it as the limit of the discrete-time discrete-valued random walk described in the previous handout, with step size $h$. Let $Z_t = \sum_{i=1}^{\lfloor t/h \rfloor} hY_n$, where $Y_n = 2X_n - 1$ and $X_1, X_2, \ldots$ be iid Bernoulli with $\theta = 1/2$ and $h > 0$. More precisely, the time interval $[0, t]$ is divided to $n$ segments of length $\delta$ and $Z_t$ counts the variables $Y_n$ multiplied by a step size $h > 0$. Recall (from previous lecture) that $E(Y_n) = E(2X_n - 1) = 2\theta - 1 = 0$ and $\text{Var}(Y_n) = \text{Var}(2X_n - 1) = 4\text{Var}(X_n) = 4\theta(1 - \theta) = 1$.

As a result we have $E(hY_n) = 0$ and $\text{Var}(hY_n) = h^2$.

We consider the above process in the limit $\delta \to 0, h \to 0, h = \sqrt{\alpha \delta}$ (for some $\alpha > 0$). In other words, the length of the segment goes to zero, the step size $h$ goes to 0 while maintaining $h = \sqrt{\alpha \delta}$.

The Wiener process $Z_t$ may be written as

$$Z_t = \lim_{\delta \to 0, h \to 0, h = \sqrt{\alpha \delta}} \sum_{i=1}^{n} hY_n = \lim_{n \to \infty} \sqrt{\alpha \delta} \sum_{i=1}^{n} Y_n = \lim_{n \to \infty} \sqrt{\alpha \delta \sqrt{t/n}} \sum_{i=1}^{n} Y_n$$

$$= \lim_{n \to \infty} \sqrt{\alpha \delta} \frac{\sum_{i=1}^{n} Y_n}{\sqrt{n}} = \sqrt{\alpha \delta} \lim_{n \to \infty} \frac{\sum_{i=1}^{n} Y_n}{\sqrt{n}}$$
where the third inequality comes from the fact that since the segment $[0, t]$ is divided to $n$ segments of length $\delta$, $n = \lceil t/\delta \rceil$ and $\delta = t/n$ (this is not an entirely rigorous argument - since we ignore the floor function - but we leave it at that.) Finally, by the central limit theorem $\lim_{n \to \infty} \frac{\sum_{i=1}^{n} Y_i}{\sqrt{n}}$ approaches a normal RV with 0 mean and 1 variance. Since $Z_t$ is $\sqrt{\alpha t}$ times that normal RV, we have that $Z_t$ is a normal RV with mean 0 and variance $\alpha t$.

So far we showed that $Z_t$ is a uni-variate normal RV. To show that it fulfills the above definition for the Wiener process we need to show (1) that it is a Gaussian process and (2) derive its auto-covariance function.

Consider the above derivation of $Z_t$ as a random walk. Since $Z_t$ is a counting process, its increments are independent and have the same distribution: i.e. $f_{Z_{t_2} - Z_{t_1}} = f_{Z_2}$. As a result, the pdf of a finite dimensional marginal $f_{Z_{t_1}, \ldots, Z_{t_k}}$ has multivariate Gaussian distribution which is a product of Gaussians

$$f_{Z_{t_1}, \ldots, Z_{t_k}}(Z) = f_{Z_{t_1}}(Z_1) f_{Z_{t_2} - Z_{t_1}}(Z_2 - Z_1) \cdots f_{Z_{t_k} - Z_{t_{k-1}}}(Z_k - Z_{k-1}).$$

The mean function $m_Z(t) = 0$ as it is the sum of zero mean RVs. The auto-covariance is

$$C_Z(t, s) = \mathbb{E}(\left(\lim_{\delta \to 0} \sum_{i=1}^{\lceil t/\delta \rceil} hY_i \right) \left(\lim_{\delta \to 0} \sum_{i=1}^{\lceil s/\delta \rceil} hY_i \right)) = \lim h^2 \mathbb{E}\left(\sum_{i=1}^{\lceil t/\delta \rceil} \sum_{j=1}^{\lceil s/\delta \rceil} Y_i Y_j \right) = \lim h^2 \sum_{i=1}^{\lceil t/\delta \rceil} \sum_{j=1}^{\lceil s/\delta \rceil} \mathbb{E}(Y_i Y_j).$$

In the above sum, $\mathbb{E}(Y_i Y_j) = 0$ for $i \neq j$ (since $Y_i$, $Y_j$ are independent with mean 0) and the auto-covariance equals

$$C_Z(t, s) = \lim h^2 \sum_{i=1}^{\min(s, t)/\delta} \mathbb{E}(Y_i^2) = \lim h^2 \min(s, t) \text{Var}(Y)/\delta = \lim \alpha \delta \min(s, t)/\delta = \alpha \min(s, t)$$

where the second equality is justified by the limit process and the third equality follows from the fact that we take the limit at $h = \sqrt{\alpha \delta}$. 

2