Some Important Random Variables

Guy Lebanon

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Recall that the cdf (for discrete or continuous RV) or pmf (for discrete RV) or pdf (for continuous RV) characterize probabilities involving the RV X such as $P(X \in A)$. For this reason, we will define RVs through these functions and not by defining the sample space Ω, the probabilities on events $P(A), A \subset \Omega$, and the RV X as a function $X : \Omega \to \mathbb{R}$.

Discrete Random Variables

The Bernoulli Trial RV

The simplest discrete RV is perhaps the Bernoulli trial RV. It has the following pmf $p_X(0) = 1-\theta$, $p_X(1) = \theta$, $0 \le \theta \le 1$ and $p_X(x) = 0$ for all other values of x. θ is said to be the parameter of the distribution. The Bernoulli trial RV may be used to characterize the probability that a (possibly biased) coin falls on heads ($X = 0$) or tails ($X = 1$). Sometime X is interpreted as an experiment or trial (with two possible outcomes) that may either fail $X = 0$ or succeed $X = 1$ with probabilities $1 - \theta, \theta$.

The Geometric RV

Imaging now that we have a sequence of independent Bernoulli trials $\{Z_i\}_{i=0}^{\infty}$ with parameter θ. The geometric RV X, counts the number of failures ($Z_i = 0$) before we encounter a success ($Z_j = 1$). The pmf is $p_X(x) = \theta(1-\theta)^x$ for $x \in \mathbb{N} = \{0, 1, \ldots\}$ and 0 otherwise. Indeed, as expected, we have

$$\sum_{m=0}^{\infty} p_X(m) = \theta(1 + (1-\theta) + (1-\theta)^2 + \cdots) = \theta \frac{1}{1-(1-\theta)} = 1.$$

The Binomial RV

The Binomial RV X counts the number of successes in n independent Bernoulli experiments with parameter θ (without ordering). If we kept the ordering, then the required probability for x successes would be $\theta^x(1-\theta)^{n-x}$. Since we disregard the ordering we need to count all possible outcomes leading to x successes (there are n choose x such outcomes). The pmf is therefore

$$p_X(x) = \binom{n}{x} \theta^x (1-\theta)^{n-x}$$

for $x = 0, 1, \ldots, n$ and $p_X(x) = 0$ otherwise. The fact that $\sum_{x=0}^{n} p_X(x) = 1$ may be ascertained by the binomial theorem

$$1 = 1^n = (\theta + (1-\theta))^n = \sum_{k=0}^{n} \binom{n}{k} \theta^k (1-\theta)^{n-k}$$

The Poisson RV

The pmf of the Poisson RV X is

$$p_X(x) = \frac{\alpha^x e^{-\alpha}}{x!}.$$

for $x \in \mathbb{N} = \{0, 1, \ldots\}$ and $p_X(x) = 0$ otherwise. Here as well

$$\sum_{k=0}^{\infty} \frac{\alpha^k e^{-\alpha}}{k!} = e^{-\alpha} \sum_k \frac{\alpha^k}{k!} = e^{-\alpha} e^{\alpha} = 1.$$

The Poisson RV is often used to count the number of occurrences of an event at a particular region (e.g. cars arriving at an intersection or phone calls arriving at a switchboard). The parameter α is the average number of events occurring. An interesting result: the binomial RV pmf approaches the Poisson pmf when $n \to \infty, \theta \to 0$ but $n\theta$ stays constant: $n\theta = \alpha$.

Continuous Random Variables

The Uniform distribution

The simplest continuous RV is perhaps the uniform RV on the interval $[a, b]$. It has the pdf $f_X(x) = 1/(b-a)$ for $a \le x \le b$ and 0 otherwise. Clearly $\int_{-\infty}^{\infty} f_X(x)dx = \frac{b-a}{b-a} = 1$. Note that the pdf is positive and constant between a and b and therefore any result $\{X = c\}, a \le c \le b$ is equally likely (recall the intuition of pdf: $f_X(x)\Delta \approx P(x < X < x + \Delta)$ as $0 < \Delta \to 0$).

The Exponential RV

The exponential RV with parameter $\lambda > 0$ has the pdf $f_X(x) = \lambda e^{-\lambda x}$ for $x > 0$ and 0 otherwise. The cdf is 0 for $x \le 0$ and for $x > 0$ it is

$$F_X(x) = P(X \le x) = \int_0^x \lambda e^{-\lambda x} = -e^{-\lambda x}\Big|_0^x = 1 - e^{-\lambda x}.$$

Note that the pdf decreases exponentially fast as x grows (for positive x) - therefore it is more probably that X will receive a small value. This RV is often used to model radioactive decay of particles. It is the unique continuous RV with the memoryless property:

$$P(X > t + h | X > t) = \frac{P(\{X > t + h\} \cap \{X > t\})}{P(X > t)} = \frac{P(\{X > t + h\})}{P(X > t)} = \frac{e^{-\lambda(t+h)}}{e^{-\lambda t}} = e^{-\lambda h} = P(X > h).$$

The Normal or Gaussian RV

The normal or Gaussian RV X with parameters $\mu \in \mathbb{R}$ and $\sigma > 0$ has the following pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

The parameter μ is the average value, and σ denotes the spread (we will be more specific later on). The cdf of the normal distribution does not have a closed form. It may be expressed through the function Φ - the cdf of a normal distribution with parameters $\mu = 0, \sigma = 1$

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} \exp\left(-\frac{(x'-\mu)^2}{2\sigma^2}\right) dx' = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(x-\mu)/\sigma} e^{-t^2/2} dt = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

where we used the change of variables $t = (x' - \mu)/\sigma$. As a byproduct we get the useful result that the if X is a Gaussian RV with some μ, σ, then $Y = g(X) = (X - \mu)/\sigma$ is a Gaussian RV with $\mu = 0, \sigma = 1$. Y is called a standard normal RV.