Examples of Probability Measures

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1 Finite Sample Spaces

If the sample space is finite $\Omega = \{\omega_1, \ldots, \omega_n\}$, it is relatively straightforward to define probability measures by defining probabilities of elementary events:

Definition 1. *For a finite sample space Ω, a set that contains only one element $E = \{\omega_1\}$ is called an elementary event.*

Suppose that we are given a set of n non-negative numbers $\{p_i\}_{i=1}^n$ that sum to one: $\sum_{i=1}^n p_i = 1$. Then there exists a unique probability distribution P over events of Ω such that $P(\{\omega_i\}) = p_i$. This probability is defined on arbitrary events $E = \{\omega_i : i \in I\}$ through the third axiom

$$P(E) = \sum_{i \in I} P(\{\omega_i\}) = \sum_{i \in I} p_i.$$

1.1 Classical Interpretation on Finite Sample Space

In the classical interpretation of probability on finite sample spaces, the probability of all elementary events $\{\omega_i\}$ are equal. Since the probability must satisfy $P(\Omega) = 1$ we have that for all ω_i, $P(\{\omega_i\}) = \frac{1}{|\Omega|}$. In general, the probability of sets E in the classical model (if the sample space is finite) is $P(E) = \frac{|E|}{|\Omega|}$.

As an example consider the experiment of throwing two distinct dice and observing the result (with order). The sample space is $\Omega = \{(x, y) : x, y \in \{1, 2, \ldots, 6\}\}$. The probability of the elementary event $E = \{(4, 4)\}$ is therefore $P(E) = \frac{1}{|\Omega|} = 1/36$. The probability of getting a sum of 9 in both dice is

$$P(\text{sum} = 9) = P(\{(6, 3), (3, 6), (4, 5), (5, 4)\}) = \frac{|\{(6, 3), (3, 6), (4, 5), (5, 4)\}|}{36} = \frac{4}{36}.$$

A more interesting example is the following: suppose in the U.S. Senate there are 60 male senators and 40 female senators. Under the classical probability assumption, what is the probability of selecting a committee of 3 senators that are all males? Recall that all elementary events have equal probability and therefore the probability is

$$\begin{aligned} P(E) = \frac{|E|}{|\Omega|} &= \frac{\text{number of ways to sample 3 out of 60 males, without order and without replacement}}{\text{number of ways to sample 3 out of 100 senators, without order and without replacement}} \\ &= \frac{\binom{60}{3}}{\binom{100}{3}} = \frac{60 \cdot 59 \cdot 58}{3 \cdot 2} \frac{3 \cdot 2}{100 \cdot 99 \cdot 98} \approx 0.211. \end{aligned}$$

Therefore, if the frequency of all-male committees is significantly larger than 21% of the entire number of committees, there is reason to believe that the classical model is not correct in this case.

2 Continuous Sample Space

In a continuous sample space, defining a probability distribution is more complicated. We will learn the general way of doing so later in the course. For now we examine one way of assigning probabilities to events in a continuous sample space that is compatible with the classical interpretation of probability.

2.1 Classical Interpretation on Continuous Sample Space

Suppose that $\Omega \subset \mathbb{R}^k$. Since events are subsets of Ω, they are also subsets of the Euclidean space. We can define the probability of an event E to be the n-dimensional volume of E divided by the n-dimensional volume of Ω.

For example, if our experiment is measuring weight of people in a particular geographical region the sample space could be $\Omega = (0, 1000) \subset \mathbb{R}^1$. The probability of the getting a measurement between 150 and 250, *in the classical model* is the ratio of the 1-dimensional volumes or lengths: $\frac{|250-150|}{|1000-0|} = 0.1$. Of course such a classical model is not likely to be useful.

If we throw darts at a board of radius 1, and assume that we never miss the board, the probability of hitting the bulls eye (a circular region of radius 0.1) under the classical model is the ratio of the areas $\frac{\pi 0.1^2}{\pi 1^2} = 0.01$. Of course such a model makes the weird assumption that the person throwing the darts does not make any attempt to hit the center.

An interesting consequence of the classical model on continuous spaces is that the probability of an elementary event is zero - since the volume of one point is 0.