The Multinomial and Multivariate Normal Distributions

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January 6, 2006

The two most important vector RVs are the multinomial (discrete) and the multivariate normal (continuous).

The Multinomial Distribution

Definition 1. *The vector RV* $\vec{\boldsymbol{X}} = (X_1, \ldots, X_n)$ *has a multinomial distribution with parameters* $N \in \{1, 2, \ldots\}$ *(note that* N *is not necessarily* n*) and* $(\theta_1, \ldots, \theta_n)$ *where* $\theta_i \geq 0$ *for all* i *and* $\sum_{i=1}^n \theta_i = 1$ *if it has the following pmf*

$$p_{\vec{\boldsymbol{X}}}(x_1, \ldots, x_n) = \begin{cases} \binom{N}{x_1, \ldots, x_n} \theta_1^{x_1} \cdots \theta_n^{x_n} & \textit{if } x_1, \ldots, x_n \textit{ are non-negative integers that sum to } N \\ 0 & \textit{otherwise} \end{cases}.$$

Here $\binom{N}{x_1, \ldots, x_n} = \frac{N!}{x_1! \cdots x_n!}$ *is the multinomial coefficient.*

The multinomial distribution applies when we have a random experiment with n possible results, each occurring with probability θ_i. The experiment is repeated N times and $X_1, \ldots, X_n$ measure the number of times the different outcomes occurred. Since there are N experiment the total number of outcomes has to be $x_1 + \cdots + x_n = N$ and since θ_i are the probability of getting outcome i in one experiment, $\sum_i \theta_i = 1$.

To see why the pmf follows from the above description consider $p_{\vec{\boldsymbol{X}}}(x_1, \ldots, x_n)$ which is the probability of getting x_1 times outcome 1, and so on until x_n times outcome n in a series of N independent experiments. $p_{\vec{\boldsymbol{X}}}(x_1, \ldots, x_n)$ is $\theta_1^{x_1} \cdots \theta_n^{x_n}$ (which is the probability of an ordered sequence of outcomes with the necessary property - x_1 times result 1 and so on) times the number of ways to obtain ordered sequences of x_1 times outcome 1 etc. That number is precisely the multinomial coefficient

$$= \binom{N}{x_1} \binom{N - x_1}{x_2} \binom{N - x_1 - x_2}{x_3} \cdots \binom{x_n}{x_n} = \frac{N!}{x_1!(N - x_1)!} \frac{(N - x_1)!}{x_2!(N - x_1 - x_2)!} \frac{(N - x_1 - x_2)!}{x_3!(N - x_1 - x_2 - x_3)!} \cdots \frac{1}{1}$$
$$= \frac{N!}{x_1! \cdots x_n!} = \binom{N}{x_1, \ldots, x_n}$$

Example: The roulette has 38 possible outcomes, 18 red, 18 black and 2 green. Thus playing the roulette is an experiment with $\theta_1 = \theta_2 = 18/38$ and $\theta_3 = 2/38$. If we play the roulette 10 times, the probability that we get 4 red outcomes, 2 black outcomes and 4 green is

$$p_{X_1, X_2, X_3}(4, 2, 4) = \frac{10!}{4!2!4!} (18/38)^4 (18/38)^2 (2/38)^4$$

The multinomial coefficient is present since there are $\frac{10!}{4!2!4!}$ ways to play 10 times and obtain 4 red 2 black and 4 green outcomes.

The Multivariate Normal Distribution

Definition 2. *The vector RV* $\vec{\boldsymbol{X}} = (X_1, \ldots, X_n)$ *has the multivariate normal distribution with parameters* $\vec{\boldsymbol{\mu}} \in \mathbb{R}^n$ *and* Σ *(a symmetric matrix of size* $n \times n$ *with positive eigenvalues) if it has the following pdf*

$$f_{\vec{\boldsymbol{X}}}(x_1, \ldots, x_n) = \frac{1}{(2\pi)^{n/2}\sqrt{\det \Sigma}} e^{-\frac{1}{2}(\vec{\boldsymbol{x}}-\vec{\boldsymbol{\mu}})^\top \Sigma^{-1}(\vec{\boldsymbol{x}}-\vec{\boldsymbol{\mu}})}$$

Here, $(\vec{\boldsymbol{x}} - \vec{\boldsymbol{\mu}})^\top$ *is the transpose of the column vector* $(\vec{\boldsymbol{x}} - \vec{\boldsymbol{\mu}})$ *and* Σ^{-1} *is the inverse of the matrix* Σ.

Since the determinant of a matrix with all positive eigenvalues is positive - there is no problem with taking its square root. The term in the exponent may be written in scalar form as:

$$-\frac{1}{2}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{\mu}})^\top \Sigma^{-1}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{\mu}}) = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - \mu_i)[\Sigma^{-1}]_{ij}(x_j - \mu_j)$$

In a way similar to the one-dimensional normal RV, the vector μ is a vector of expectations $\mathsf{E}(X_i) = \mu_i$ and the matrix Σ is the matrix of covariances and variances

$$[\Sigma]_{ij} = \begin{cases} \mathsf{Var}(X_i) & i = j \\ \mathsf{Cov}(X_i, X_j) & i \neq j \end{cases}$$

Several important special cases:

1. If Σ is the identity matrix, its determinant is 1, its inverse is the identity as well, and the exponent becomes $-\sum_{i=1}^{n}(x_i - \mu_i)/2$ which indicates that the pdf factors into the product of n pdf functions of normal RV, with means μ_i and variance 1 $\sigma_i^2 = 1$. Thus in this case, the multivariate normal vector RV is just independent $X_1, \ldots, X_n$ - each of which is normal with parameters $\mu_i, 1$.

2. If Σ is diagonal matrix with elements $[\Sigma]_{ij} = \delta_{ij}\sigma_i^2$, then its inverse is a diagonal matrix with elements $[\Sigma^{-1}]_{ij} = \delta_{ij}(1/\sigma_i^2)$ and its determinant is $\det \Sigma$ is the product of the diagonal elements $\prod_i \sigma_i^2$. Again, the term in the exponent of the pdf factors into a sum which indicates that the pdf factors into a product of marginal pdfs for each of the variables X_i. Thus, again we have that $X_1, \ldots, X_n$ are independent normal RV with parameters (μ_i, σ_i^2) (verify!).

3. In the general case, the shape of the pdf (its contour levels) are determined by the exponent (since the term $\frac{1}{(2\pi)^{n/2}\sqrt{\det \Sigma}}$ is constant as a function of $\vec{\boldsymbol{x}}$) which is a quadratic form $-\sum_i \sum_j (x_i - \mu_i)\Sigma_{ij}^{-1}(x_j - \mu_j)$. As a result, the contour levels of the pdf will be ellipse shape - with the center of the ellipse determined by $\vec{\boldsymbol{\mu}}$ and the shape determined by Σ^{-1}. If Σ^{-1} is the identity, the ellipse will be spherical. If Σ^{-1} is diagonal with different elements on the diagonal the ellipse shape will be axis aligned with the spread in different axes determined by the diagonal elements.

As a consequence of (2) above we see that if $X_1, \ldots, X_n$ are multivariate normal, and have 0 covariance, then Σ and Σ^{-1} are diagonal and $X_1, \ldots, X_n$ are independent. This is in contrast to the statement that *in general*, covariance 0 does not imply independence.