

Go is PSPACE hard

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Clique Talk

“Go is Polynomial-Space Hard” David
Lichtenstein and Michael Sipser.

Some Complexity Reminders

- PSPACE is the class of languages that are decidable in polynomial space on a deterministic Turing machine
- $NL \subseteq P \subseteq NP \subseteq PSPACE = NPSPACE \subseteq EXPTIME$
- $P \subset EXPTIME$
- $NL \subset PSPACE$

TQBF

- Boolean formulas with quantifiers are *quantified Boolean formulas*
 - $\phi_1 = \forall x \exists y [(x \vee y) \wedge (-x \vee -y)]$ is true
 - $\phi_2 = \exists x \forall y [(x \vee y) \wedge (-x \vee -y)]$ is false
- When each variable falls within the scope of some quantifier, the formula is *fully quantified*
- The TQBF problem is to determine whether a fully quantified Boolean formula is true or false
- **Theorem:** *TQBF is PSPACE complete*

Plan of Action

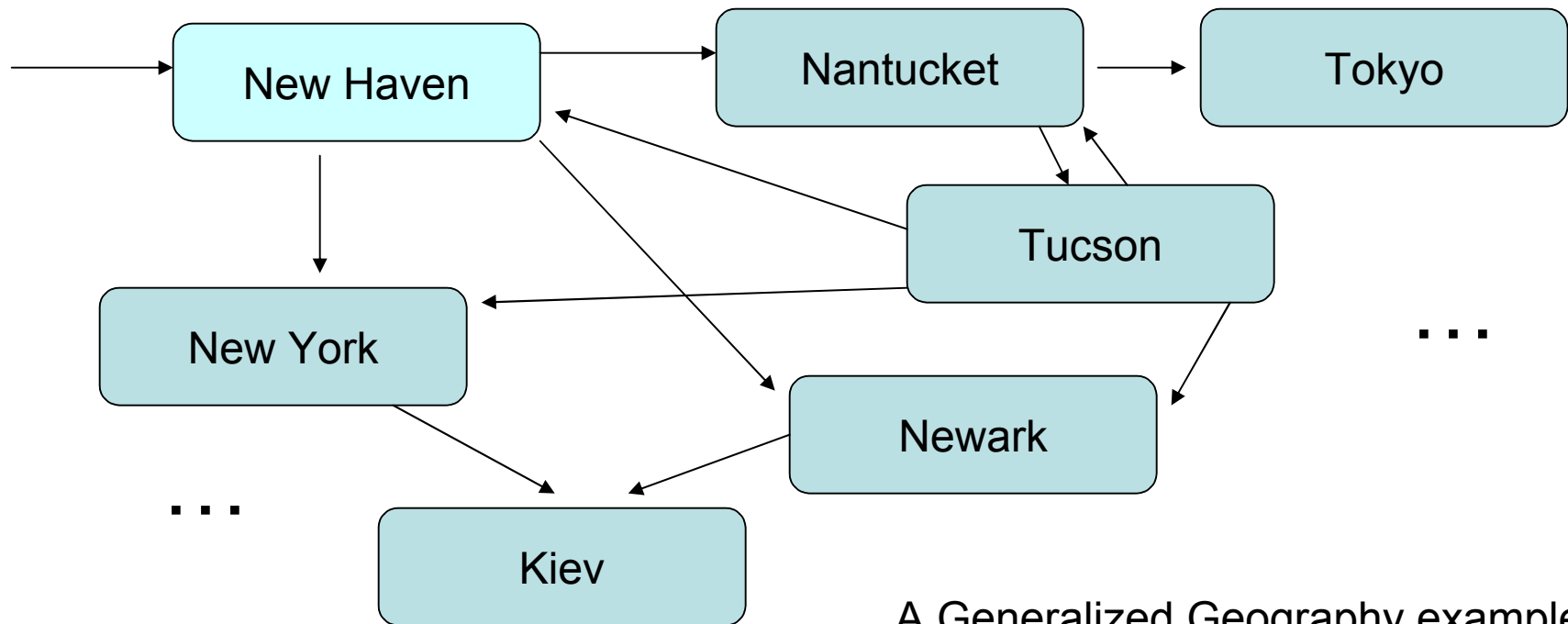
- Reduce TQBF to Generalized Geography
- Reduce Generalized Geography to Planar Generalized Geography
- Teach you the Rules of Go
- Reduce Planar Geography to Go

Generalized Geography (GG)

- 2 player game in which players take turns naming cities from anywhere in the world.
- Each city chosen must begin with the same letter that ended with the previous city's name.
- Repetition isn't permitted.
- Game starts with some designated city and ends when a player cannot continue.

GG Continued

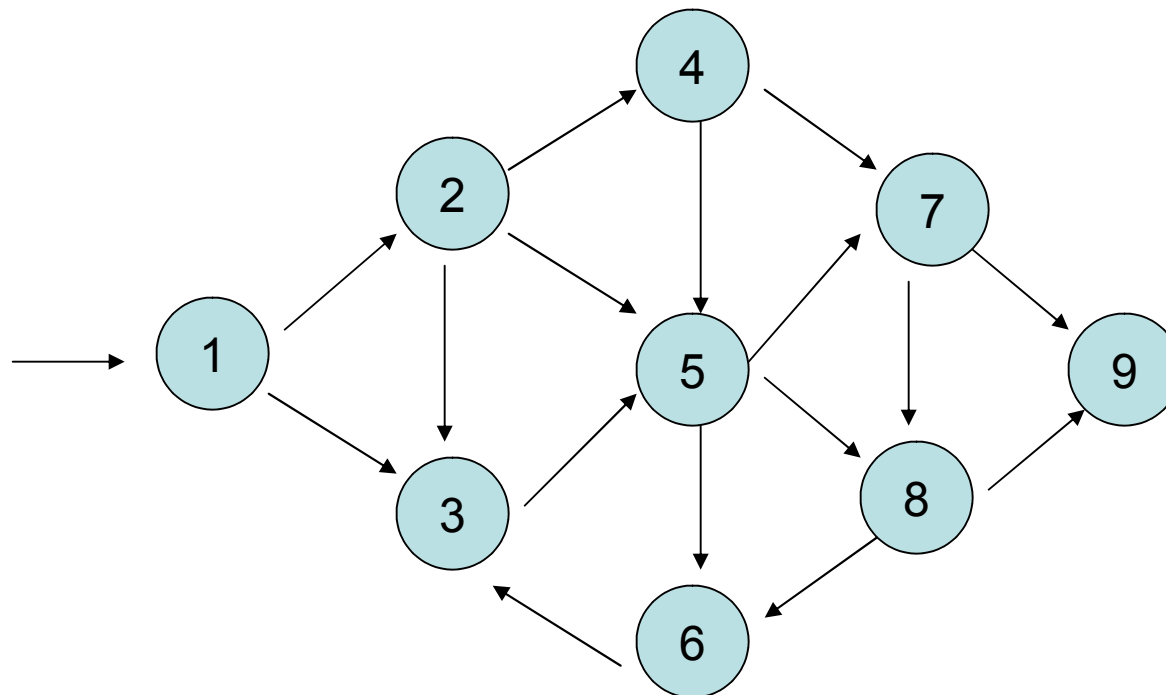
- **Theorem:** *The problem of determining which player has a winning strategy in a generalized geography game is PSPACE-complete.*



A Generalized Geography example

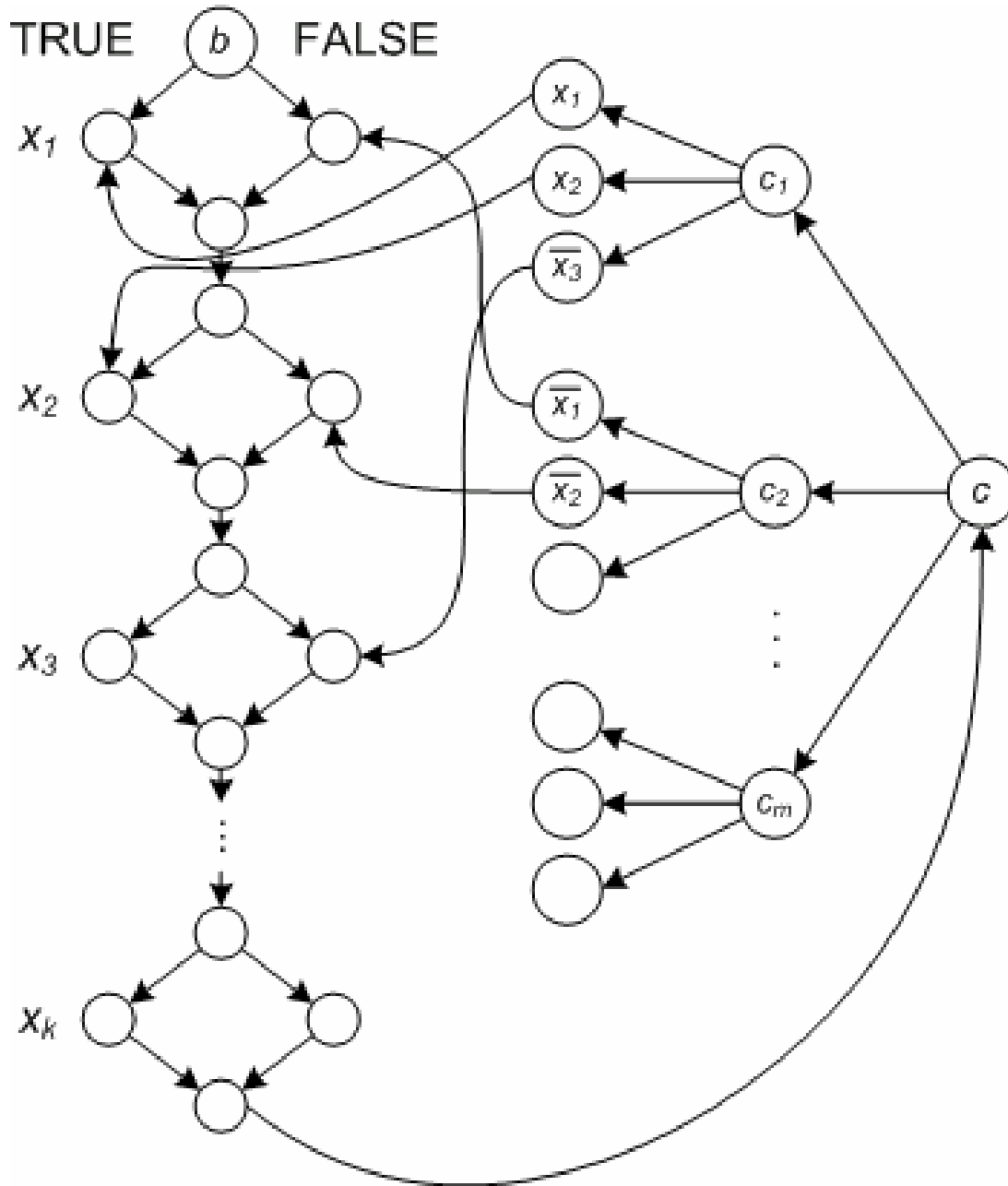
A Sample Game

- Any directed graph with a designated start node is an example of generalized geography



Who wins?

GG is PSPACE hard



A FQBF can be expressed in the form:
 $\phi = \exists x_1 \forall x_2 \exists x_3 \dots [\psi]$
where ψ is in CNF

Player 1 is E, player 2 is A
If ϕ is true, player E wins.

Play starts at b w/ E
At c, it's A's turn

If ϕ is false, player A wins
by choosing unsat clause.
Else, player E can win by
selecting a sat variable.

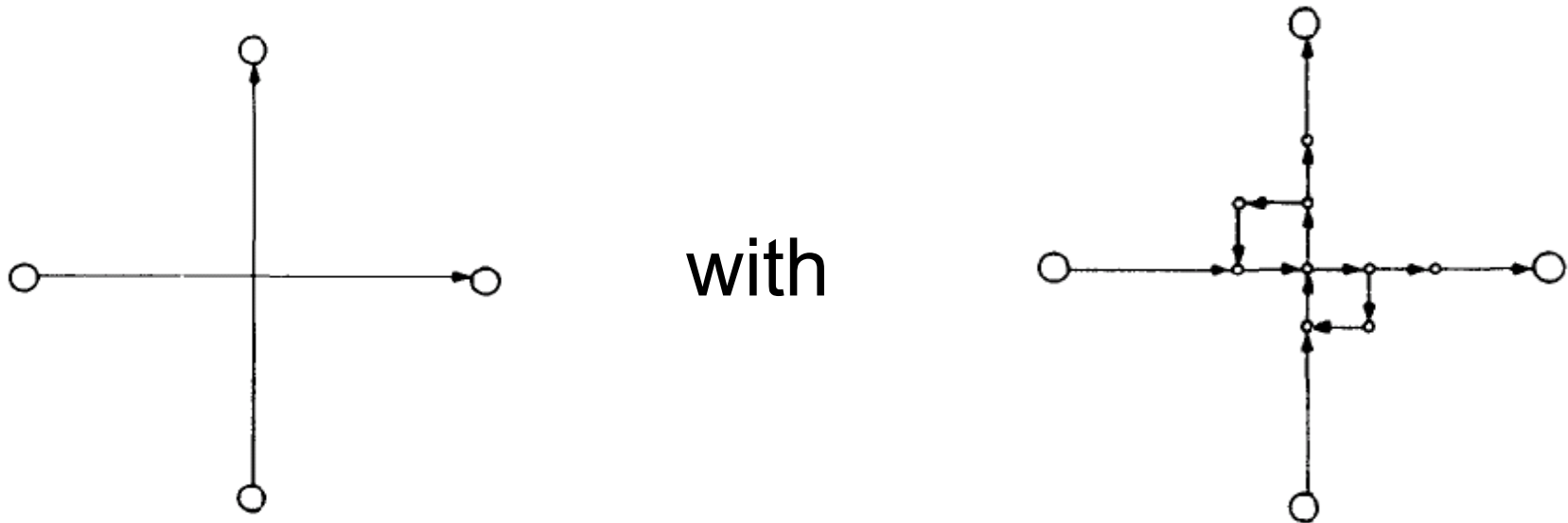
QED.

Progress Check

- Reduce TQBF to Generalized Geography
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Planar GG is PSPACE hard

- Planar Generalized Geography is GG played on planar graphs
- Draw GG in the plane, allow arcs to cross
- Just replace and QED:



with

Simple case analysis shows this works

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The Rules of Go (from [LS '80])

- Go is played on a grid of $n \times n$ locations called points
- There are 2 players: black and white. Black moves first.
- A player moves by placing a stone of his color on a vacant point or passes.
- The game terminates when both players pass consecutively

More Rules

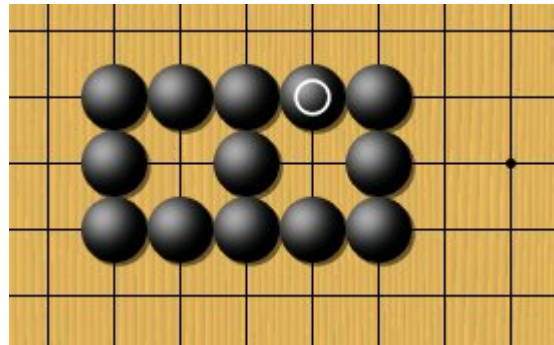
- As the game progresses, stones form clusters called groups
- A group is a maximal contiguous region of uniformly colored stones
- A group of stones becomes surrounded if none of them is adjacent to a vacant stone
- After each black move, all surrounded white stones are removed, followed by all surrounded black stones. (and vice versa)

Scoring

- At the end of the game all dead stones are removed from the board
 - A stone is dead if it ultimately can be surrounded despite any attempts to save it
- Then, a vacant point is white territory if it is surrounded on all sides by either white stones or an edge of the board. (or black)
- The final score for white is the count of white territory plus the number of black stones removed from the board at any time. (and vice versa)
- The player with the highest score wins.

Eyes

- “Two Eyes you’re alive”



- Frequently, in a game, a player may have a nearly surrounded group of stones which he is desperately trying to connect to a group with two eyes. His opponent is trying to cut him off. The proof exploits this situation.

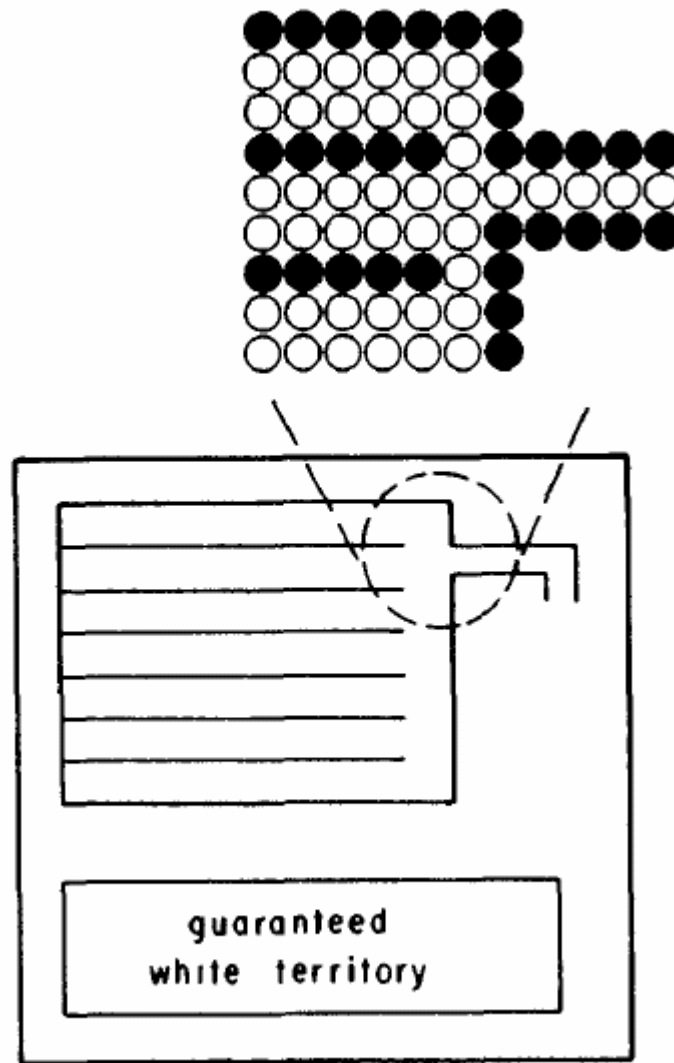
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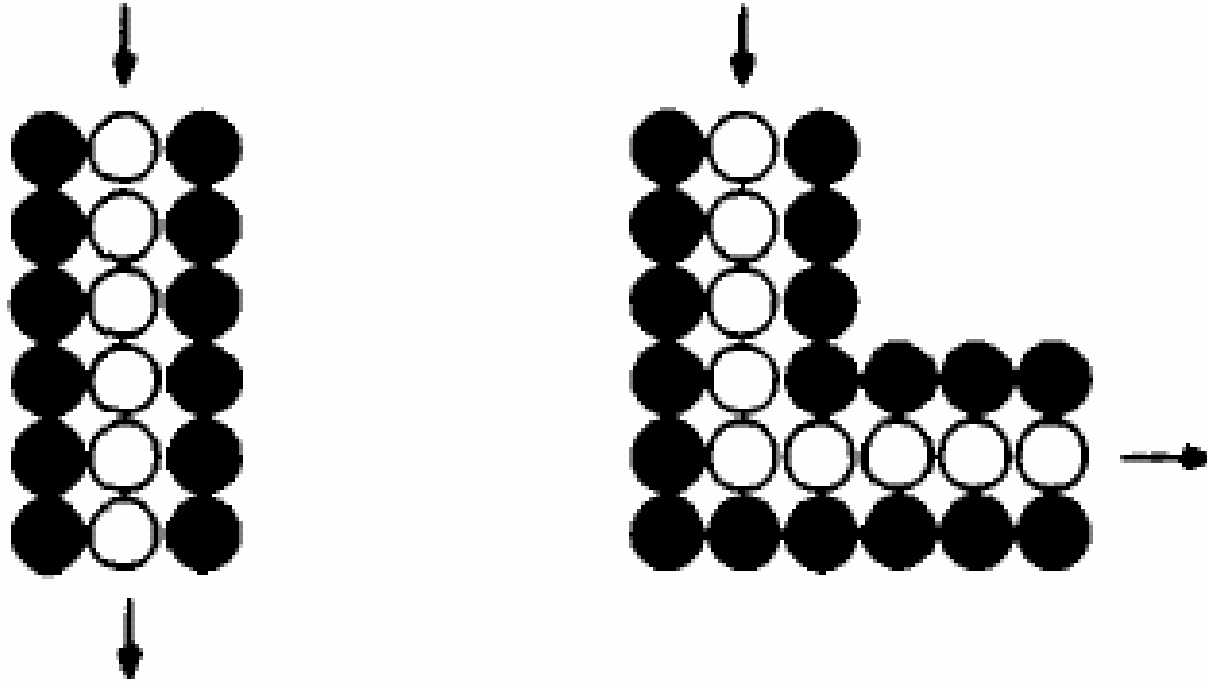
The Reduction

- We will make a Go position that will have the property that Black has a winning strategy only if Planar GG player E does.
- Proof will make use of the following ideas:
 - white will have a lot of territory
 - white will have an even larger group trying to escape capture through a “breach” in blacks wall
 - outcome of the game will hinge on whether black can take the white group
 - the breach leads to a planar GG structure
 - B and W will be forced to play a geography game

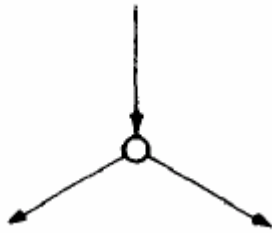
The Global Picture



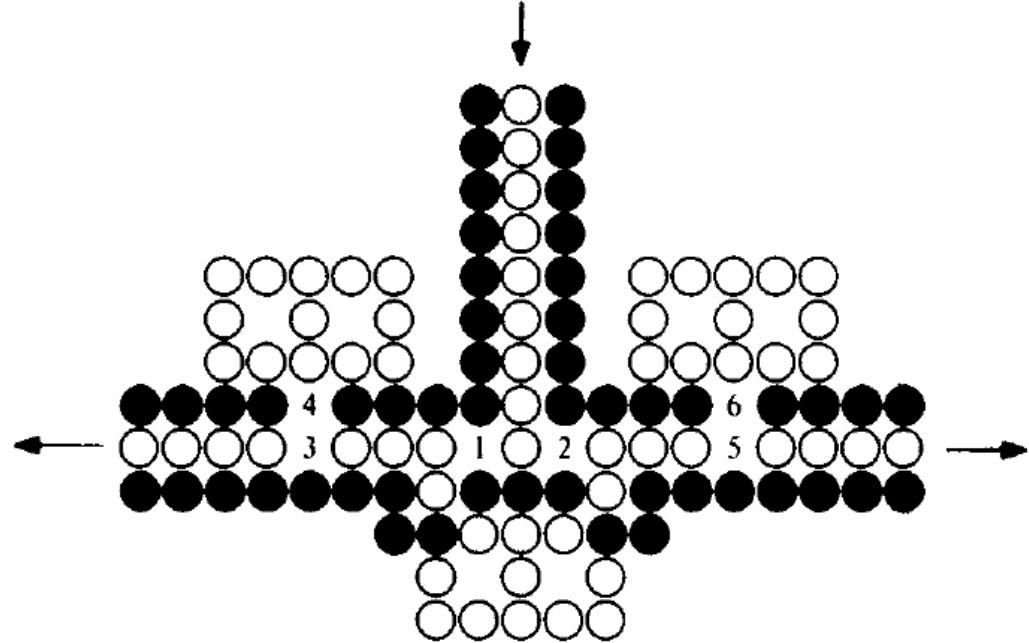
Widgets: Edges



Widget: Vertex type 1



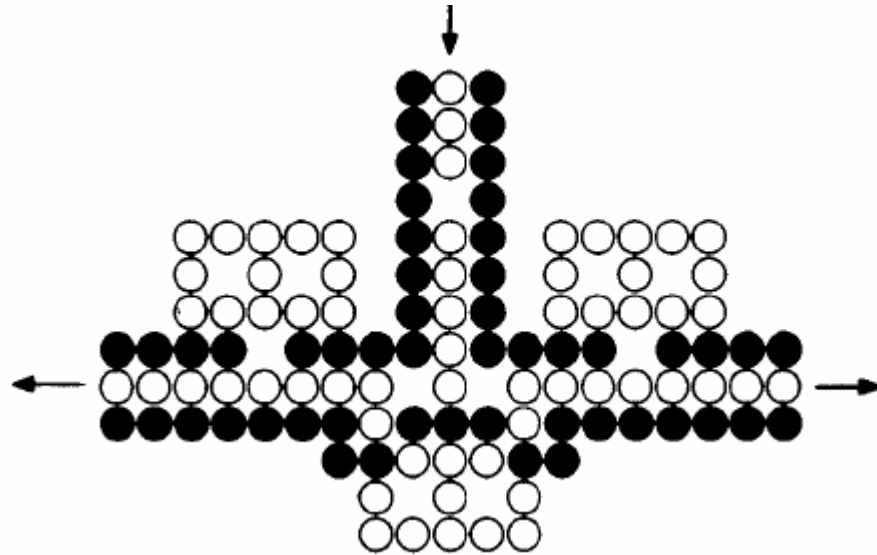
Player A's choice



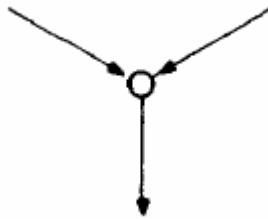
Widget: Vertex type 2



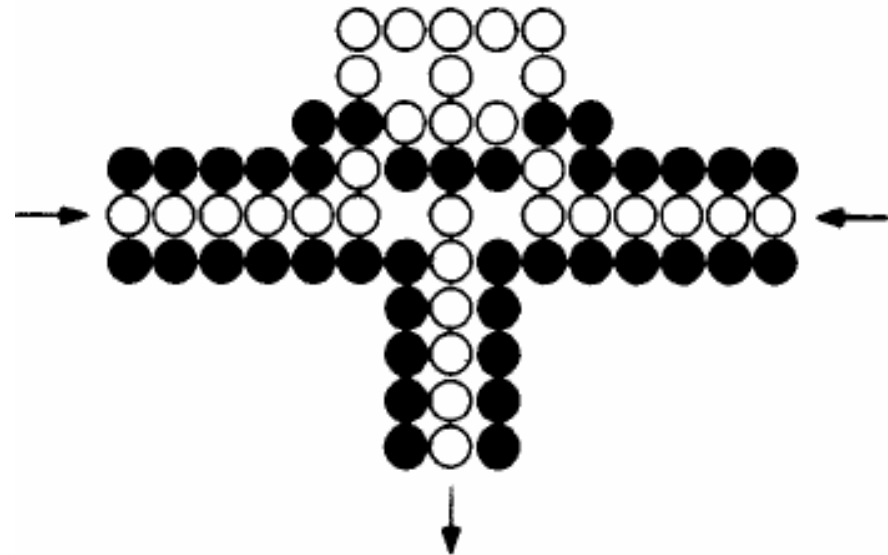
Player E's choice



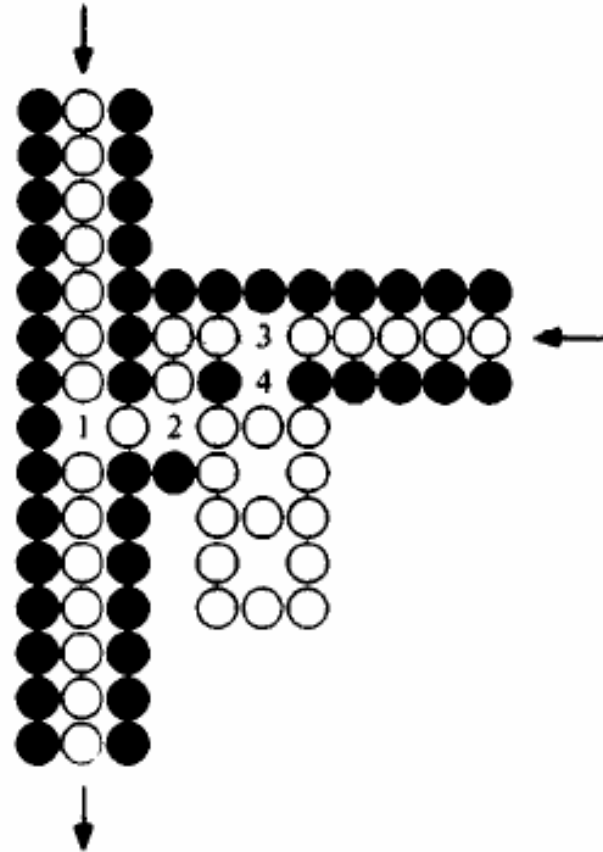
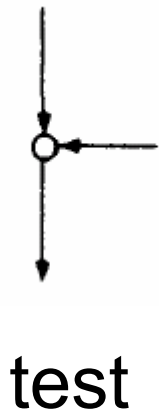
Widget: Vertex type 3



join



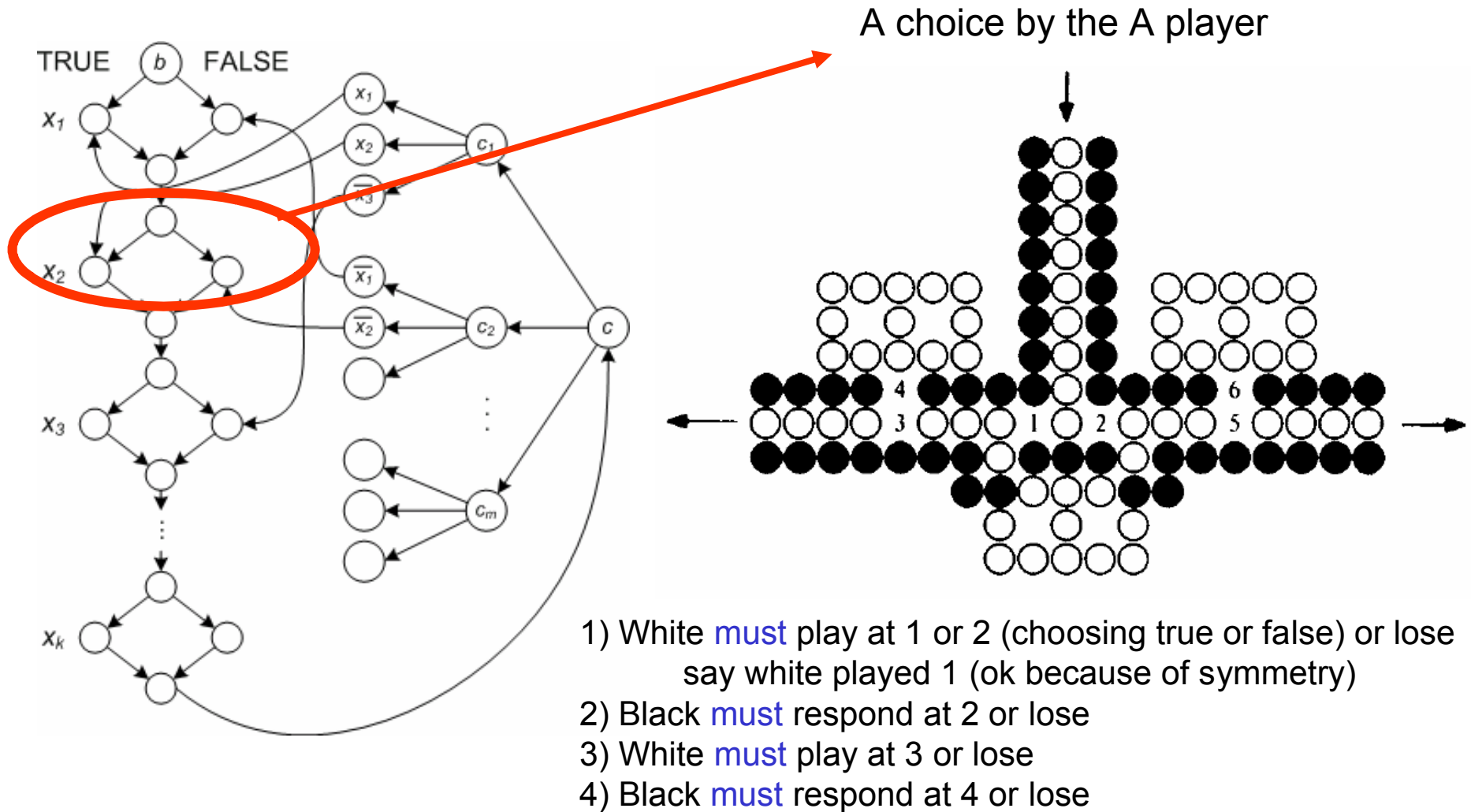
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The Game Play (review)

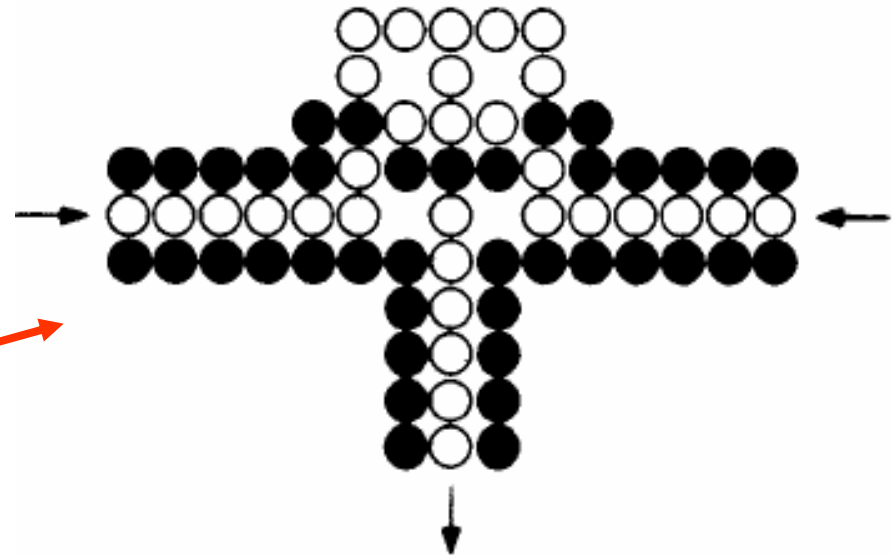
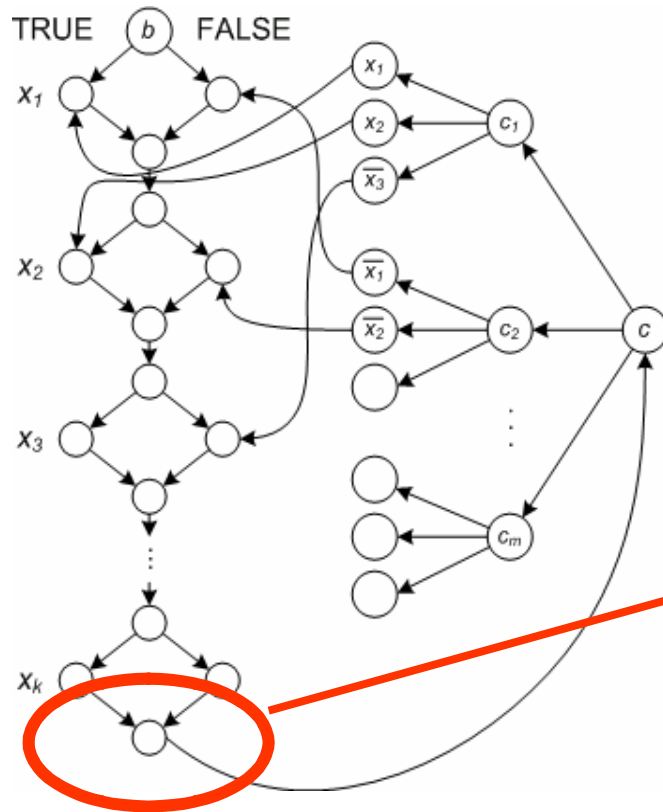
- We connect the go configurations to make the planar GG out of Go widgets.
- White and Black make choices by playing go moves
- In the end – either white survives or is taken. If white survives – then player “A” wins (the original FQBF was not satisfiable). Else black survives.

A Close Look

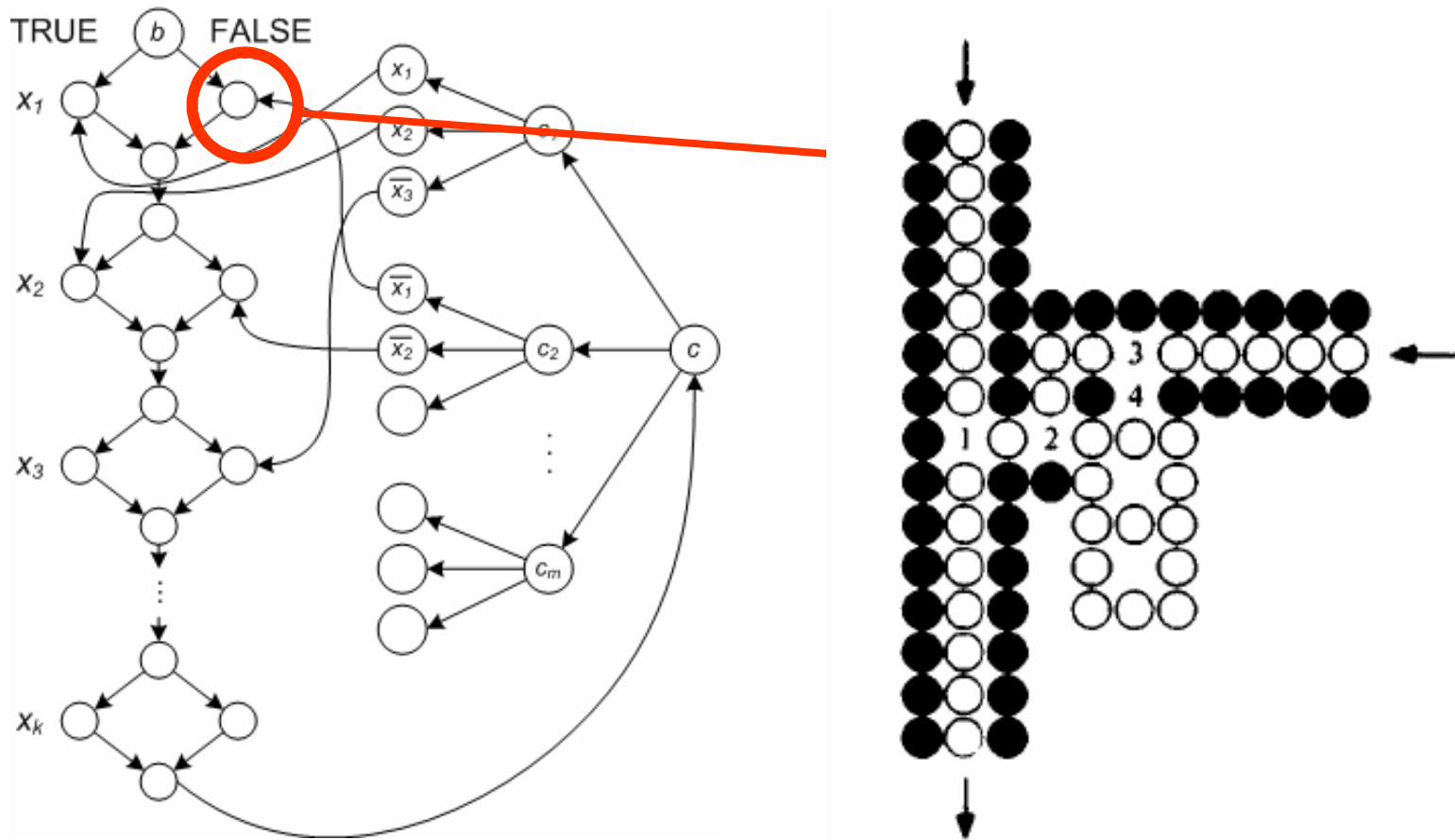


A Close Look Continued

- The choice for player E is similar argument, only black chooses sequence.
- The choice for the “join” is obvious

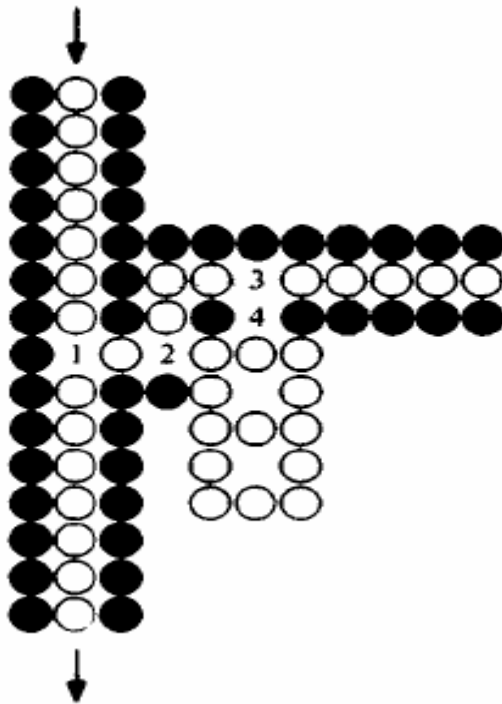


The Test



The Test

- We show that if play enters through right hand pipe, black wins iff play previously passed through the vertical pipe.



- 1) If play already passed through junction on top, it will have left on bottom.
- 2) If play subsequently enters through the right-hand pipe, Black wins.
- 3) If play enters through right hand pipe first, then White wins.

QED

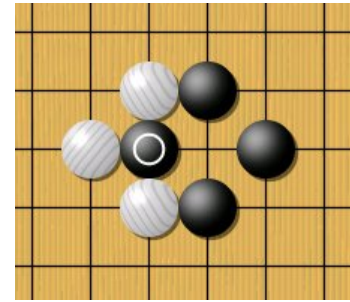
- We have reduced planar GG to Go. Black wins this game if player E would have won the GG (Which would happen only if TQBF was true), Showing Go is NP hard!
- This means that looking at a Go position, it's (probably) hard to tell who will win. In fact, it's PSPACE hard.
- I cheated on a little part of the proof – did anyone catch me?

Just One More Thing...

- Reduce TQBF to Generalized Geography
- Reduce Generalized Geography to Planar Generalized Geography
- Teach you the Rules of Go
- Reduce Planar Geography to Go
- Confuse you a little more

Ko, Super Ko, and EXPTIME

- There is another rule to the game of Go - called Ko. It leads to Ko-fights. Some rule sets also have Super Ko rules.



- With Ko, Go is EXPTIME-complete.
- *As far as I know*, Go (without Ko) has neither been shown to be in PSPACE nor has it been shown to be EXPTIME-complete.
- With super-Ko, what is the complexity of Go?

References

- D. Lichtenstein and M. Sipser, Go is hard, *J. ACM* 27 (1980) 393-401. polynomial-space
- J. M. Robson, The complexity of Go, *Proc. IFIP* (1983) 413-417.
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- M. Crășmaru and J. Tromp, Ladders are PSPACE-complete, *Proc. 2nd Int. Conf. Computers and Games*, Springer-Verlag, 2000, pp. 241-249.