Lecture 1, 08-22-11
CS 2050, Intro Discrete Math for Computer Science

\( S_n = 1+2+3+\ldots+n = ? \)

**Note:** Recall that for the above sum we can also use the notation \( S_n = \sum_{i=1}^{n} i \).

We will use a direct argument, in this case from first principles, to express \( S_n \) in closed form.

In general, a direct proof is a reduction to a broader and well known general truth, or a straightforward combination of already established facts without making any further assumptions.

It is also called proof by placement, meaning "placement" to the well known general truth.

It also goes by the Latin name **modus ponens**. This is short for "modus ponendo ponens " which literally means "the way that affirms by affirming".

**Theorem.** \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \), \( \forall n \in \mathbb{N} \).

**Note:** Recall that \( \forall \) means for all and \( \mathbb{N} \) is the set of natural numbers \( 1, 2, 3, \ldots \) to infinity.

**Proof.** Let \( S_n = \sum_{i=1}^{n} i \).

Consider the \( n \times n \) board, whose area consists of \( n^2 \) squares.

Realize that \( S_n \) is the area below (and including) the diagonal (leftmost picture):

1 square from the first line, 2 squares from the second line, etc, \( n \) squares from the last line.

Similarly, \( S_n \) is also the area above (and including) the diagonal (middle picture).

If we sum these two areas (rightmost picture) we get the area of the entire board, with the diagonal counted twice. We may thus write:

\[
S_n + S_n - n = n^2 \quad \iff \quad 2S_n = n^2 + n \quad \iff \quad 2S_n = n(n+1) \quad \iff \quad S_n = \frac{n(n+1)}{2} .
\]

**Note:** Recall that \( \iff \) means if and only if or implies both ways.

\( X \iff Y \) means either both \( X \) and \( Y \) are true, or both \( X \) and \( Y \) are false.

This completes the proof of the theorem.

\( \square \)
Theorem. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, $\forall n \in \mathbb{N}$.

Proof. Let $S_n = \sum_{i=1}^{n} i$.

Consider an $n \times n$ matrix, with $n^2$ entries indexed $(i, j)$, for $1 \leq i \leq n$ and $1 \leq j \leq n$.

Thus,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} 1 = n^2$$  \hspace{1cm} (1)

Note: When we have an $n \times n$ matrix, it is very useful to think in terms of the indices $(i, j)$ of its entries.

We can now express the sum in (1) in terms of $S_n$ as follows:

Note: In the set of equalities below, make sure that you understand the use and meaning of the $\sum$ and corresponding ( ) notations.

$$\sum_{i=1}^{n} \sum_{j=1}^{n} 1 = \sum_{i=1}^{n} \left( \left( \sum_{j=1}^{i} 1 \right) + \left( \sum_{j=i}^{n} 1 \right) - 1 \right)$$

$$= \left( \sum_{i=1}^{n} \sum_{j=1}^{i} 1 \right) + \left( \sum_{i=1}^{n} \sum_{j=i}^{n} 1 \right) - \left( \sum_{i=1}^{n} 1 \right)$$

$$= \left( \sum_{i=1}^{n} i \right) + \left( \sum_{j=i}^{n} \sum_{i=1}^{j} 1 \right) - n$$

$$= \left( \sum_{i=1}^{n} i \right) + \left( \sum_{j=i}^{n} j \right) - n$$

$$= 2S_n - n$$  \hspace{1cm} (2)

Combining (1) and (2) immediately implies

$$n^2 = 2S_n - n$$

Thus

$$S_n = \frac{n(n+1)}{2}$$

This completes the proof of the theorem.
In Recitation, Wed 08-24-11, the following sums were also discussed:

\[ S_{\text{odd}} = 1 + 3 + 5 + 7 + \ldots + (2n - 1) = ? \]

\[ S_{\text{even}} = 2 + 4 + 6 + 8 + \ldots + 2n = ? \]

By inspection of Figures A and B below we may immediately infer that
\( S_{\text{odd}} \) is the area of an \( n \times n \) square, thus \( S_{\text{odd}} = n^2 \),
\( S_{\text{even}} \) is the area of an \( n \times (n + 1) \) rectangle, thus \( S_{\text{even}} = n(n + 1) \).

![Figure A](image1.png)

![Figure B](image2.png)

If one did not observe the correspondance with areas of squares and rectangles, the above sums can be also obtained by reduction to the first Theorem concerning \( 1 + 2 + \ldots + n = \frac{n(n+1)}{2} \).
Theorem. $\sum_{i=1}^{n}(2i-1) = n^2$, $\forall n \in \mathbb{N}$.

Proof.

\[
\begin{align*}
\sum_{i=1}^{n}(2i-1) &= \left(\sum_{i=1}^{n}2i\right) - \left(\sum_{i=1}^{n}1\right) \\
&= \left(2\sum_{i=1}^{n}i\right) - n \\
&= 2\frac{n(n+1)}{2} - n, \text{ by the Theorem of page 1} \\
&= n(n+1) - n \\
&= n(n+1-1) \\
&= n \times n \\
&= n^2.
\end{align*}
\]

This completes the proof of the theorem.

\[\square\]

Theorem. $\sum_{i=1}^{n}2i = n(n+1)$, $\forall n \in \mathbb{N}$.

Proof.

\[
\begin{align*}
\sum_{i=1}^{n}2i &= 2\sum_{i=1}^{n}i \\
&= 2\frac{n(n+1)}{2}, \text{ by the Theorem of page 1} \\
&= n(n+1).
\end{align*}
\]

This completes the proof of the theorem.

\[\square\]
Theorem 1. We can tile the $8 \times 8$ board using $2 \times 1$ tiles.

Proof. We can tile each row laying 4 tiles horizontally from left to right. (Alternatively, we can tile each column laying 4 tiles vertically from top to bottom.)

This completes the proof of the Theorem 1.

□

Theorem 2. We cannot tile the $8 \times 8$ board with one corner removed using $2 \times 1$ tiles.

Proof. Each tile covers 2 squares. However the $8 \times 8$ board with one corner removed has $8 \times 8 - 1 = 63$ squares. This is an odd number of squares. Since each tile covers 2 squares, every possible placement of tiles covers an even number of squares. Therefore, there is no placement of tiles that can cover 63 squares.

This completes the proof of the Theorem 2.

□

Theorem 3. We can tile the $8 \times 8$ board with the two top corners removed using $2 \times 1$ tiles.

Proof. The top row has 6 squares. We can therefore tile the first row laying 3 tiles horizontally from left to right. We can further tile all the other rows laying 4 tiles horizontally from left to right.

This completes the proof of the Theorem 3.

□
Theorem 4. We cannot tile the $8 \times 8$ board with two opposite corners removed using $2 \times 1$ tiles.

Proof. Let us color the squares of the $8 \times 8$ board black and white, so that no two horizontally or vertically consecutive squares have the same color (like a chessboard.) This results in 32 black and 32 white squares in the complete board, with opposite corners colored with the same color. Therefore, when we remove two opposite corners from the board, we are left with either 30 black and 32 white squares, or with 30 white and 32 black squares. Each tile covers 2 squares of opposite colors, one black and one white. Therefore, any placement of tiles covers the same number of black and white squares. However, the $8 \times 8$ board with two opposite corners removed does not have the same number of black and white squares. Therefore, no placement of tiles $2 \times 1$ tiles can cover all 62 squares.

This completes the proof of the Theorem 4.

Where did the idea of chessboard-like coloring come from ??