Problem 1: a problem of the following type

Testing: Elementary direct proofs, proofs by contradiction, notion of counter-example, and understanding of "if then" (a.k.a. \(\Rightarrow\)) versus "if and only if" (a.k.a. \(\iff\)).

(a) Prove that if an integer \(n\) is odd then \(n^2\) is also odd.
(b) Prove that, if \(n\) is an integer and \(n^2\) is odd, then \(n\) is odd.
(c) Prove that if the sum of two integers is odd then one of them is even and the other one is odd.

Problem 2: a problem of the following type

Testing: Understanding of \(\sum\) and \(\prod\) notation and ability to use the basic fact \(\sum_{i=1}^{n} i = \frac{n(n+1)}{2}\).

(a) Prove that, for all positive integers \(n\), \(\sum_{i=1}^{n} (4i-2i) = n(n+1)\).
(b) Prove that, for all positive integers \(n\), \(\sum_{i=1}^{n} ((i+1)^2-i^2-1) = n(n+1)\).
(c) Prove that, for all positive integers \(n\) that are multiples of 3, \(\sum_{i=\frac{n}{3}}^{n} i = \frac{1}{3}n(4n+3)\).
(d) Prove that, for all positive integers \(n\), \(\prod_{i=1}^{n} 4^i = 2^{n(n+1)}\).

Problem 3: a problem of the following type

Testing: Ability to use the basic fact that, for \(x \neq 0\), \(\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}\) and understanding of log notation and elementary properties.

(a) Prove that, for all positive integers \(n\) that are powers of 3, \(\sum_{i=\log_3 n}^{n} 3^i < \frac{3^2}{2}\).
(b) Prove that, for all positive integers \(n\) that are powers of 3, \(\sum_{i=\log_3 n}^{n} 9^i = \frac{2n^2-1}{8}\).

Problem 4: a problem concerning GCDs and Euclid’s Algorithm

(a) Compute gcd(138,82), using Euclid’s algorithm. Show your work.
(b) Give the prime factorization of 138 and 82, and use this factorization to verify your answer in part (a).

Problem 5: a problem of the type

Testing: Modular arithmetic, upto and including the material in Homework 3 but not fast exponentiation and not the Chinese Remainder Theorem.

(a) Find \(x\) in the range \(\{0, 1, 2\}\) such that \(7x \equiv 2 \mod 3\).
(b) Find \(x\) in the range \(\{0, 1, 2, 3, 4\}\) such that \(1004x \equiv 1001 \mod 5\).