Problem 1
Let \( n \) and \( m \) be integers.
(a) Prove that if at least one of \( n \) or \( m \) is even then the product \( n \times m \) is even.
(b) Prove that if both \( n \) and \( m \) are odd then the product \( n \times m \) is odd.

Problem 2
(a) Let \( S_n = \sum_{i=1}^{n} i \). Prove that, for every \( n \geq 2 \), \( S_n > n \).
(b) Prove that there is unique positive integer that equals the sum of the positive integers not exceeding it.

Problem 3
Let \( S_n \) be the sum of all positive integers from 1 to \( n \), ie \( S_n = 1 + 2 + \ldots + n \) or \( S_n = \sum_{i=1}^{n} i \).
Let \( S'_n \) be the sum of the squares of all positive integers from 1 to \( n \), ie \( S'_n = 1^2 + 2^2 + \ldots + n^2 \) or \( S'_n = \sum_{i=1}^{n} i^2 \).
Let
\[
S''_n = 1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + n(n+1)
= \sum_{i=1}^{n} i(i+1)
\]
Prove that \( S''_n = S_n + S'_n \).

Problem 4
Prove that, for every positive integer \( n \), \( \sum_{i=1}^{2n} (1 + (-1)^i) = 2n \).

Problem 5
(a) Prove that the 5 \( \times \) 5 board with the top left corner removed can be covered using 2 \( \times \) 1 tiles.
(b) Prove that, for any odd integers \( n > 1 \) and \( m > 1 \), the \( n \times m \) board with the top left corner removed can be covered using 2 \( \times \) 1 tiles.

Problem 6: Extra Credit
Let \( n > 1 \) and \( m > 1 \) be odd integers. Let \( S \) be the set of squares of the \( n \times m \) board. Give a complete characterization of the (single) squares that, if removed from the \( n \times m \) board, then the remaining \((n \times m) - 1\) area can be covered using 2\( \times \)1 tiles. That is, characterize the set \( T \subseteq S \) such that
\[
x \in T \implies S \setminus \{x\} \text{ can be covered using } 2 \times 1 \text{ tiles}
\]
\[
x \in S \setminus T \implies S \setminus \{x\} \text{ cannot be covered using } 2 \times 1 \text{ tiles}
\]