Problem 1: 5 Points
Argue that, for every positive integer \( n \), and for all \( x \) such that \( 0 < x < 1 \),
\[
\sum_{k=0}^{n} x^k < \frac{1}{1 - x}.
\]

Hint: Class lecture notes from 08-31-11 (on the web).

Answer
We know that, for every positive integer \( n \), and for all \( x \) such that \( x \neq 1 \),
\[
\sum_{k=0}^{n} x^k = \frac{x^{n+1} - 1}{x - 1} = \frac{1 - x^{n+1}}{1 - x}.
\]

When \( 0 < x < 1 \), \( 0 < x^{n+1} < 1 \),
thus \( (1 - x^{n+1}) > 0 \) and \( (1 - x) > 0 \), so it is more convenient to use:
\[
\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}.
\]

We may now immediately see:
\[
\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}.
\]

\[
= \frac{1}{1 - x} - \frac{x^{n+1}}{1 - x}
\]

\[
< \frac{1}{1 - x} \quad \text{since } x^{n+1} > 0 \text{ and } (1 - x) > 0
\]

thus \( \frac{x^{n+1}}{1-x} > 0 \).
Problem 2: 10 Points
Argue that, for every positive integer \( n \), and for all \( x \) such that \( 0 < x < 1 \),
\[
\sum_{k=0}^{n} k x^k < \frac{1}{(1-x)^2}.
\]
Hint: Class lecture notes from 08-31-11 (on the web).

Answer
We know that, for every positive integer \( n \), and for all \( x \) such that \( x \neq 1 \),
\[
\sum_{k=0}^{n} k x^k = \frac{(n+1)x^{n+2} - (n+2)x^{n+1} + 1}{(x-1)^2}
\]
\[
= \frac{(n+1)x^{n+2} - (n+2)x^{n+1} + 1}{(1-x)^2}
\]
\[
= \frac{1}{(1-x)^2} - \frac{(n+2)x^{n+1} - (n+1)x^{n+2}}{(1-x)^2}
\]
\[
= \frac{1}{(1-x)^2} - \frac{x^{n+1}}{(1-x)^2} (n+2 - x(n+1))
\]
\[
= \frac{1}{(1-x)^2} - \frac{x^{n+1}}{(1-x)^2} (n(1-x) + (2-x))
\]
\[
< \frac{1}{(1-x)^2}.
\]
The last inequality follows because, for \( 0 < x < 1 \), \( (1-x) > 0 \) and \( (2-x) > 0 \), hence
\[
n(1-x) + (2-x) > 0
\]
and
\[
\frac{x^{n+1}}{(1-x)^2} (n(1-x) + (2-x)) > 0.
\]
Problem 3: 5 Points
Argue that, for every positive integer \( n \), \( \sum_{i=0}^{n} (2^{i+1} - 2^i) = 2^{n+1} - 1 \).

Answer

\[
\sum_{i=0}^{n} (2^{i+1} - 2^i) = \sum_{i=0}^{n} (2 \times 2^i - 2^i) = \sum_{i=0}^{n} (2^i + 2^i - 2^i) = \sum_{i=0}^{n} 2^i = \frac{2^{n+1} - 1}{2 - 1} \text{ since, for all } x \neq 1, \sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x-1} = 2^{n+1} - 1 .
\]
Problem 4: 5 Points

Argue that, for every positive integer \( n \), \( \sum_{i=0}^{n} (3^i - 2^i) = \frac{1}{2} (3^{n+1} - 2^{n+2} + 1) \).

Answer

\[
\sum_{i=0}^{n} (3^i - 2^i) = \sum_{i=0}^{n} 3^i - \sum_{i=0}^{n} 2^i \\
= \frac{3^{n+1} - 1}{3 - 1} - \frac{2^{n+1} - 1}{2 - 1} \quad \text{since, } \forall x \neq 1, \sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x - 1} \\
= \frac{3^{n+1} - 1}{2} - (2^{n+1} - 1) \\
= \frac{3^{n+1} - 1}{2} - \frac{2^{n+2} - 2}{2} \\
= \frac{1}{2} (3^{n+1} - 1 - 2^{n+2} + 2) \\
= \frac{1}{2} (3^{n+1} - 2^{n+2} + 1) .
\]
Problem 5: 5 Points
Argue that, for every positive integer $n$ that is a power of 2,

$$\sum_{k=0}^{\log_2 n} 2^k = 2n - 1.$$ 

Recall: Elementary property of the log function: $a^{\log_a x} = x$.

Answer

$$\sum_{k=0}^{\log_2 n} 2^k = \frac{2^{(\log_2 n)+1} - 1}{2 - 1} \quad \text{since, } \forall x \neq 1, \sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x - 1}$$

$$= 2 \times 2^{\log_2 n} - 1$$

$$= 2n - 1 \quad \text{since } a^{\log_a x} = x \text{ thus } 2^{\log_2 n} = n.$$
Problem 6: 10 Points
Argue that, for every positive integer $n$ that is a power of 2,

$$
\sum_{k=0}^{\log_2 n-1} \left( n \left( \frac{3}{2} \right)^k \right) = 2\left(n^{\log_2 3} - n \right).
$$

Recall: Elementary properties of the log function: $a^{\log_a x} = x$ and $\log_a x = \frac{\log_b x}{\log_b a}$.

Answer

$$
\sum_{k=0}^{\log_2 n-1} \left( n \left( \frac{3}{2} \right)^k \right) = n \sum_{k=0}^{\log_2 n-1} \left( \frac{3}{2} \right)^k
$$

$$
= n \left( \frac{\left( \frac{3}{2} \right)^{\log_2 n} - 1}{\frac{3}{2} - 1} \right)
\text{ since, } \forall x \neq 1, \sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x-1}
$$

$$
= n \left( \frac{\frac{3^{\log_2 n}}{2^{\log_2 n}} - 1}{\frac{3}{2} - 1} \right)
$$

$$
= 2n \left( \frac{3^{\log_2 n}}{2^{\log_2 n}} - 1 \right)
$$

$$
= 2n \left( \frac{3^{\log_2 n}}{n} - 1 \right) \text{ since } 2^{\log_2 n} = n
$$

$$
= 2 \left( n \frac{3^{\log_2 n}}{n} - n \right)
$$

$$
= 2 \left( 3^{\log_2 n} - n \right)
$$

$$
= 2 \left( (2^{\log_2 3})^{\log_2 n} - n \right) \text{ since } 3 = 2^{\log_2 3}
$$

$$
= 2 \left( 2^{\log_2 3 \times \log_2 n} - n \right)
$$

$$
= 2 \left( 2^{\log_2 n \times \log_2 3} - n \right) \text{ since } 2^{\log_2 n} = n
$$

$$
= 2 \left( n^{\log_2 3} - n \right).
$$