Problem 2, question (a)

Theorem \(2n+1 < 2^n\), for all integers \(n \geq 3\)

Proof Inductive on \(n\)

Base Case For \(n=3\) \(2 \cdot 3 + 1 = 7\), \(2^3 = 8\)
thus \(2 \cdot 3 + 1 < 2^3\) is true.

Inductive Hypothesis Assume that for \(n = k \geq 3\), \(2k + 1 < 2^k\).

Inductive Step We want to show that, for \(n = k+1\), \(2(k+1) + 1 < 2^{k+1}\).

But \(2(k+1) + 1 = 2k + 2 + 1\)
\[= (2k + 1) + 2\]
\[< 2^k + 2\], applying the inductive hypothesis \(2k + 1 < 2^k\)

\[< 2^k + 2^k\], using \(2 < 2^k\) for \(k \geq 3\)
\[= 2 \cdot 2^k\]
\[= 2^{k+1}\]

\(\therefore\) QED
Problem 2, Question (b)

**Theorem** \( n^2 < 2^n \), for all integers \( n > 5 \)

**Proof** inductive on \( n \).

**Base Case** For \( n = 5 \), \( 5^2 = 25 \), \( 2^5 = 32 \);
thus \( 5^2 < 2^5 \) is true.

**Inductive Hypothesis** For \( n = k \geq 5 \), assume that \( k^2 < 2^k \).

**Inductive Step** We want to show that, for \( n = k + 1 \), \( (k+1)^2 < 2^{k+1} \).

But \( (k+1)^2 = k^2 + 2k + 1 \)
\[ < k^2 + (2k+1) \]
\[ < 2^k + 2^k \]
\[ = 2 \times 2^k \]
\[ = 2^{k+1} \]

Applying the inductive hypothesis \( k^2 < 2^k \)
Applying the Theorem \( 2n+1 < 2^n \), for all integers \( n \): proved in part (a) of this problem.

Q.E.D.
Problem 3, part (a)

Theorem 3 divides \( n^3 + 2n \), for all positive integers \( n \geq 3 \)

Proof inductive on \( n \).

Base case: we need to verify that, for \( n = 1 \),
\[
3^3 + 2 \cdot 1 = 1 + 2 = 3
\]
which is obviously a multiple of 3.

Inductive Hypothesis For \( n = k \geq 3 \), assume that there exists an integer \( q \) such that \( k^3 + 2k = 3q \).

Inductive Step For \( n = k + 1 \), we need to show that there exists an integer \( q' \) such that
\[
(k+1)^3 + 2(k+1) = 3q'
\]
But
\[
(k+1)^3 + 2(k+1) = k^3 + 3k^2 + 3k + 2k + 2
\]
\[
= (k^3 + 2k) + (3k^2 + 3k + 3)
\]
\[
= (k^3 + 2k) + 3(k^2 + k + 1)
\]
\[
= 3q + 3(k^2 + k + 1)
\]
applying the inductive hypothesis \( k^3 + 2k = 3q \).
\[
= 3(q + k^2 + k + 1)
\]
since 3 is a common factor.
\[
= 3q' \quad \text{for} \quad q' = q + k^2 + k + 1 \quad \text{QED}
\]
Problem 3, part (b) 

Theorem \[ \sum_{i=1}^{n} i^3 = \left( \frac{n(n+1)}{2} \right)^2 \], for all integers \( n \geq 1 \).

Proof inductive on \( n \).

Base Case For \( n = 1 \), we need to verify that \[ \sum_{i=1}^{1} i^3 = \left( \frac{1(1+1)}{2} \right)^2 \]

But \[ \sum_{i=1}^{1} i^3 = 1 \] and \[ \left( \frac{1(1+1)}{2} \right)^2 = \left( \frac{1 \cdot 2}{2} \right)^2 = 1 \]

Inductive Hypothesis For \( n = K \geq 1 \) assume that \[ \sum_{i=1}^{K} i^3 = \left( \frac{K(K+1)}{2} \right)^2 \]

Inductive Step For \( n = K+1 \), we want to show that \[ \sum_{i=1}^{K+1} i^3 = \left( \frac{(K+1)(K+2)}{2} \right)^2 \]

But, \[ \sum_{i=1}^{K+1} i^3 = \left( \sum_{i=1}^{K} i^3 \right) + (K+1)^3 \]

separating out the last term of the sum

\[ \left( \frac{K(K+1)}{2} \right)^2 + (K+1)^3 \]

applying the inductive hypothesis

\[ = \frac{K^2(K+1)^2}{2^2} + \frac{4(K+1)(K+1)^2}{2^2} \]

calculations

\[ = \frac{(K+1)^2}{2^2} \left( K^2 + 4(K+1) \right) \]

taking \( \frac{(K+1)^2}{2^2} \) out as common factor

\[ = \frac{(K+1)^2}{2^2} \left( K^2 + 4K + 4 \right) = \frac{(K+1)^2(K+2)^2}{2^2} = \left( \frac{(K+1)(K+2)}{2} \right)^2 \]

QED
Problem 5

Theorem: For every integer $n \geq 8$, there exist integers $p \geq 0$ and $q \geq 0$ such that 

$$n = 3p + 5q$$

Proof: Strong induction on $n$.

Base Case: For $n = 8$, $n = 9$ and $n = 10$ we verify that:

$$8 = 3 \times 1 + 5 \times 1, \quad 9 = 3 \times 3 + 5 \times 0 \quad \text{and} \quad 10 = 3 \times 0 + 5 \times 2$$

Inductive Hypothesis: For $n = k \geq 10$, assume that:

for every integer $k'$ in the range $8 \leq k' \leq k$,

there exist integers $p' \geq 0$ and $q' \geq 0$ such that

$$k' = 3p' + 5q'$$

Inductive Step: For $n = k+1$, we need to show that:

there exist integers $p \geq 0$ and $q \geq 0$ such that

$$k+1 = 3p + 5q$$

go to the next page
We proceed to establish the inductive step as follows:

\[ k+1 = k + (-2 + 2) + 1 \]
\[ = (k-2) + (2+1) \]
\[ = (k-2) + 3 \quad \rightarrow \quad \text{notice that } k \geq 10 \Rightarrow (k-2) \geq 8 \]
\[ \Rightarrow \quad \text{thus setting } k' = (k-2) \quad \text{we have } \quad k' \geq 8 \]
\[ = k' + 3 \]
\[ = 3(p' + 5q') + 3 \quad \rightarrow \quad \text{applying the inductive hypothesis to } k' \]
\[ = 3(p' + 1) + 5q' \]
\[ = 3p + 5q \quad , \quad \text{for } \quad p = p' + 1 \]
\[ \quad \text{and} \quad q = q' \]