(to Sample Problems)  

**Problem 1**

(a)  
\[ 5^2 \mod 7 = 25 \mod 7 = 4 \]
\[ 5^4 \mod 7 = (5^2 \mod 7)(5^2 \mod 7) \mod 7 = (4 \times 4) \mod 7 = 16 \mod 7 = 2 \]
\[ 5^8 \mod 7 = ((5^4 \mod 7)(5^4 \mod 7)) \mod 7 = (2 \times 2) \mod 7 = 4 \mod 7 = 4 \]

From here on the pattern repeats:

\[ 5^{16} \mod 7 = 2 \]
\[ 5^{32} \mod 7 = 4 \]
\[ 5^{64} \mod 7 = 2 \]
\[ 5^{128} \mod 7 = 4 \]
\[ 5^{256} \mod 7 = 4 \]
\[ 5^{512} \mod 7 = 2 \]
\[ 5^{1024} \mod 7 = 4 \]

And we knew that:

\[ 4 \times 4 \mod 7 = 2 \]
\[ 2 \times 2 \mod 7 = 4 \]

(b)  
\[ 5^{301} \mod 7 = 5^{256+32+8+4+1} \mod 7 \]
\[ = 4 \times 4 \times 4 \times 2 \times 5 \mod 7 \]
\[ = 2 \times 2 \times 4 \times 5 \mod 7 \]
\[ = 2 \times 5 \mod 7 = 10 \mod 7 = 3 \]

\[ 5^{102} \mod 7 = 5^{301} \times 5^{301} \mod 7 \]
\[ = 3 \times 3 \mod 7 \]
\[ = 9 \mod 7 = 2 \]

\[ 5^{903} \mod 7 = 5^{301} \times 5^{301} \times 5^{301} \mod 7 \]
\[ = 3 \times 3 \times 3 \mod 7 \]
\[ = 27 \mod 7 = 6 \]
a) According to the definition of $O(\cdot)$, 
\[ f(x) = O(g(x)) \text{ if and only if } \exists C > 0 \quad \exists x_0 : f(x) \leq C g(x), \quad \forall x \geq x_0 \]

\[
10x^3 = 10x^3 \quad \forall x
\]

\[-100x^2 \leq 0 \quad \forall x
\]

\[+1000x \leq 1000x^3 \quad \forall x \geq 1
\]

\[-12 \leq 0
\]

\[
10x^3 - 100x^2 + 1000x - 12 < 1010x^3, \quad \forall x \geq 1
\]

Thus \[10x^3 - 100x^2 + 1000x - 12 = \Theta(x^3)\]
because it satisfies the definition with \[C = 1010\text{ and } x_0 = 1\]

b) \[
\sqrt{\log_2 n}, \quad \log_{10} n, \quad \log_2 n, \quad n^{\frac{1}{3}}, \quad \sqrt{n}, \quad n \log_2 n, \quad n^2, \quad 2^{\log_2 n}, \quad 2^n, \quad 3^n, \quad 2^{n \log_2 n}
\]

\[
= (2^{\log_2 n})^n = n^n
\]
(a) \[ 10,000 + 0.5\times 10,000 + 1,000 = 16,000 = f(0) \quad \text{ie year 2012+0} \]
\[ 16,000 + 0.5\times 16,000 + 1,000 = 25,000 = f(1) \quad \text{ie year 2012+1} \]
\[ 25,000 + 0.5\times 25,000 + 1,000 = 38,500 = f(2) \quad \text{ie year 2012+2} \]

(b) Base Case

\[ n = 0 \quad \Rightarrow \quad f(0) = 16,000 \quad \text{as computed above.} \]

and
\[ 1.5 \times 18,000 - 2,000 = 1\times 18,000 - 2,000 = 16,000 \]

Hypothesis

\[ n = k \geq 0 \quad f(k) = 1.5^k \times 18,000 - 2,000 \quad \text{is assumed to be true} \]

Induction

Step

We want to show
\[ f(k+1) = 1.5^{k+1} \times 18,000 - 2,000 \]

But by the definition of the growth of \( f \):
\[ f(k+1) = 1.5 \cdot f(k) + 1,000 \]
\[ = 1.5 \left(1.5^k \times 18,000 - 2,000\right) + 1,000 \]
\[ = 1.5^{k+1} \times 18,000 - 3,000 + 1,000 \]
\[ = 1.5^{k+1} \times 18,000 - 2,000 \]