

Every assignment will be due at the beginning of class. Recall that you can collaborate in groups and/or use external references, but you must acknowledge the group/references that you used, and you must *always write your solutions alone*. Remember that for 90% of the people, more than 50% of the understanding happens during writing/implementation/etc. (And this is not true only for CS 1050. It is true for mostly everything, at least technical).

Please read the entire homework before starting to work on it. This homework (a)revisits basic proof techniques (direct, contradiction, contrapositive) (b)refers to basic sums (c)addresses introductory aspects of recursion and induction. Please refer to the corresponding lecture outlines and assigned reading material. All are posted on the class website. You should definitely refer to the posted solutions for Homeworks 1, 2 and Quiz 1 to verify that you are complying with "appropriate" writing style for answering problems and questions. Repeat: Just as you have to write "code" that others should be able to understand relatively easily (if you are software engineers), and just as you have to write "reports", or "essays", or "opinions" that a reader should be able to easily "get the point" (if you are managers, scientists, writers, bloggers etc), similarly you have to eventually be able to write "technical arguments" so that convey to the reader relatively easily your sequence of thoughts.

Please stop by for questions during office hours of instructor or TAs and send email to mihail@cc.gatech.edu with title 1050 at any time. This helps you, but it also helps us! Sometimes it helps us understand where the class stands and where we should put more or less emphasis. And sometimes, you give us presentational and technical ideas that we would have not thought of otherwise. So keep all communication links open!

Please print this document, and write your solutions on the printout.
Please hand-in the completed printout.

PRINT YOUR NAME HERE:.....

WRITE YOUR EMAIL HERE:.....

Problem 1: (20 points)

(a) Prove that there is no positive integer n such that $n^2 + n^3 = 100$. State clearly the steps of your proof, and state clearly the proof technique(s) that you are involving in each step.

(b1) Prove that, for any integer n , if n is even, then $3n + 5$ is odd. This is a very easy proof, but you should state clearly the proof technique that you are using. For example, if you use a direct proof, you should state clearly the general principle that is used in your direct proof.

(b2) Prove that, for any integer n , if $3n + 5$ is odd, then n is even. Again, state clearly the proof technique that you are using.

(c) Prove that $\sqrt{2} + \sqrt{3}$ is irrational. State clearly the proof technique that you are using.

Problem 2: (20 points)

In class and in the book we proved that $1 + 2 + \dots + n = n(n + 1)/2$.

In class and in the book we proved that, for any $x \neq 1$, $1 + x + x^2 + \dots + x^n = \frac{x^{n+1}-1}{x-1}$.

(a1) Find a closed form for each of the sums below, and explain how you computed the closed form:

$$2 + 4 + 6 + \dots + 2n = ?$$

$$1 + 3 + 5 + \dots + (2n + 1) = ?$$

$$1 + 4 + 7 + 10 + \dots + (3n + 1) = ?$$

$$1 + 2 + 2^2 + 2^3 + \dots + 2^n = ?$$

$$1 + 2^2 + 2^4 + 2^6 + \dots + 2^{2n} = ?$$

(a2) Express each of the above sums in recursive form. Write clearly the base of the recursion.

In class we reviewed the basic properties of the log function. In particular, recall the *definition*: for any non-negative reals x and y , $\log_y x = z$ if and only if $y^z = x$.

(b1) Compute: $\log_2 4 = ?$, $\log_2 16 = ?$, $\log_2 2 = ?$, $\log_{10} 100 = ?$, $\log_{10} 10000 = ?$, $\log_2 1 = ?$, $\log_2 10 = ?$, $\log_2 \frac{1}{2} = ?$, $\log_2 0.25 = ?$, $\log_{10} 5 = ?$, $\log_{10} 0.2 = ?$, $\log_{10} 0.001 = ?$,

(b2) Using the definition of the log function, prove that:

$$\log_y y^x = x$$

$$\log_y x_1 + \log_y x_2 = \log_y (x_1 \cdot x_2)$$

$$\log_y (x^n) = n \log_y x$$

$$\log_y x = \frac{\log_z x}{\log_z y}$$

Problem 3: (20 points)

(a1) Suppose that N is any positive integer, and let a_1, a_2, \dots, a_N be a list of distinct numbers. Suppose that an algorithm finds the maximum number in the list, using comparisons only. Argue that, *any* such algorithm, *must* use *at least* $N - 1$ comparisons. State clearly the involved *invariant*.

(a2) Describe an algorithm that uses *exactly* $N - 1$ comparisons.

Suppose that N is a power of 2 (ie $N = 2^n$, for some n), and consider the following method of finding the maximum of N distinct numbers a_1, a_2, \dots, a_N . In the first round we compare a_1 with a_2 , a_3 with a_4 , \dots , a_{N-1} with a_N . Let $b_1, b_2, \dots, b_{N/2}$ be the winners. In the second round we compare b_1 with b_2 , b_3 with b_4 , \dots , $b_{\frac{N}{2}-1}$ with $b_{\frac{N}{2}}$. Let $c_1, c_2, \dots, c_{N/4}$ be the winners. And so on.

(b1) Write this algorithm recursively (pseudocode). The recursive algorithm should outline that, in order to find the maximum of N distinct elements, we find the maximum of the first $N/2$ elements, then we find the maximum of the last $N/2$ elements, and then we compare the two maximums.

(b2) Let $T(N)$ be the number of comparisons involved in the algorithm. Write a recursion for $T(N)$. Make sure you include the base case, ie what is $T(1)$.

(b3) Can you see what is the solution to this recursion? Explain, in English, a heuristic to find the solution to this recursion. (If you are ahead of the class, you may also use any formal method to solve the recursion).

Problem 4: (20 points)

(a) Prove, by induction, that for every positive integer n :

$$1 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(b) Prove, by induction on n , that for every non-negative integer n :

The solution to the recursion $T(n) = 2T(n-1) + 1$ with $T(0) = 0$ is $T(n) = 2^n - 1$.

The solution to the recursion $T(n) = T(n-1) + n$ with $T(1) = 1$ is $T(n) = n(n+1)/2$.

Problem 5: (20 points)

(a) Use mathematical induction to show that, if n people stand in a line with $n \geq 2$, and if the first person is a man and the last person is a woman, then somewhere inside the line there is a man standing right in front of a woman.

(b) Use mathematical induction to show that, for every $n \geq 0$, every complete binary tree with $N = 2^n$ leaves has exactly $N - 1$ internal (non-leaf) nodes.