

# **Basic Proof Methods**

**(using rules of inference of logic)**

**1. Direct Proofs,**  
**- including invariance in algorithms**

**2. Indirect Proofs:**  
**Contraposition and Contradiction**

**3. Hypothetical syllogism**  
**and examining cases (almost all the time...)**

**1. Direct Proofs (logical foundation: Modus Ponens)  
in practice, reduce to an obvious more general principle**

**Essentially , reduce to more general principle.**

**Sometimes identifying the more general principle  
is straightforward.**

**Sometimes identifying the more general principle is trickier,  
and we will review “strategies” for doing so.**

## 1. Direct Proofs (continued)

in practice, reduce to an obvious more general principle

**Example:** Prove that, for all real numbers  $x$ ,

$$5x^2 - 4x + 1000 \geq 0 \quad .$$

The more general principle is obvious:

The sum of non-negative quantities

is always a non-negative quantity.

So we have to express  $5x^2 - 4x + 1000$

as the sum of non-negative quantities:

$$5x^2 - 4x + 1000 =$$

$$(x^2 - 2x + 1) + (x^2 - 2x + 1) + 3x^2 + 998 =$$

$$(x - 1)^2 + (x - 1)^2 + 3x^2 + 998 \quad .$$

## 1. Direct Proofs (continued)

in practice, reduce to an obvious more general principle

**Example:** The  $n \times m$  “chocolate bar” problem:  
Any algorithm takes exactly  $n \times m - 1$  moves.

The more general principle is less obvious,  
but once you see it, it is a one line argument:

Algorithm **INVARIANT:** Any step of any algorithm  
increases the number of pieces by exactly one.

It is now trivial to complete the proof:

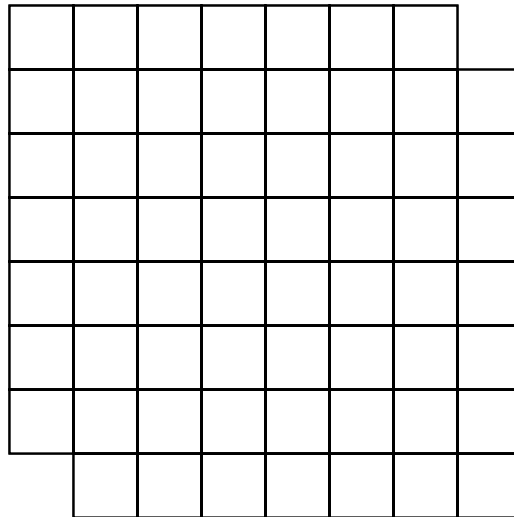
We start with 1 piece, we end-up with  $n \times m$  pieces,  
which implies that  $n \times m - 1$  moves are sufficient and necessary.

# 1. Direct Proofs (continued)

in practice, reduce to an obvious more general principle.

**Example:** Tiling the 8x8 board with 2x1 tiles.

**Claim:** Tiling the 8x8 board with 2x1 tiles is not possible, if two squares at opposite corners are removed.



The more general principle is less obvious, but once you see it, it is again a one line argument.

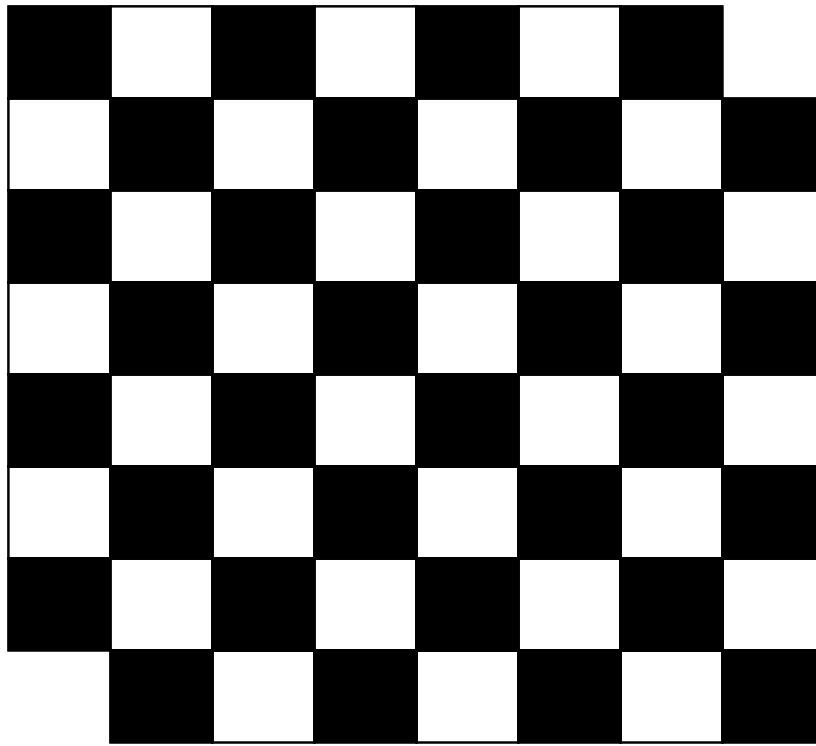
**To identify such more general principles,  
one has to look more at the structure of the problem:**

11	12	13					
21							

**To identify such more general principles,  
one has to look more at the structure of the problem:**

1+1	1+2	1+3					
2+1							

E	O	E					
O	E						



**Now the general principle is obvious:**

**INVARIANT:** Each  $2 \times 1$  tile, no matter how it is placed, reduces the number of uncovered squares by 2: one black and one white.

**We start with 30 white squares and 32 black squares.**

**Any tiling effort will end up with 2 untilted black squares, which are impossible to cover with a  $2 \times 1$  tile.**

## 2. Proofs by **Contrapositive**

**Example:** If  $n = a \times b$ , for positive integers  $n, a, b$ , then either  $a \leq \sqrt{n}$  or  $b \leq \sqrt{n}$ .

**Proof:** We will **equivalently** prove the **contrapositive**:

If, for positive integers  $n, a, b$ ,  
 $a > \sqrt{n}$  and  $b > \sqrt{n}$ ,  
then  $n \neq a \times b$ .

**Now this is obvious:**

$$a > \sqrt{n} \text{ and } b > \sqrt{n} \quad \Rightarrow$$

$$a \times b > \sqrt{n} \times \sqrt{n} = n \quad \Rightarrow$$

$$a \times b \neq n$$

### 3. Proofs by **Contradiction**

**Example:**  $\sqrt{2}$  is not a rational number.

**Proof:** Suppose, for the purposes of **contradiction**, that

$$\sqrt{2} \text{ is rational} \quad \Rightarrow$$

$$\sqrt{2} = \frac{p}{q},$$

where  $p$  and  $q$  are relatively prime integers  $\Rightarrow$

$$2q^2 = p^2,$$

where  $p$  must be even and  $q$  must be odd  $\Rightarrow$

$$2q^2 = (2k)^2 \quad \Rightarrow$$

$$2q^2 = 4k^2 \quad \Rightarrow$$

$$q^2 = 2k^2 \quad \Rightarrow$$

$q$  must be even **contradicting**

