Mixing Times of Markov Chains on 3-Orientations of Planar Triangulations

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What is a 3-orientation?

An orientation of the **internal** edges of a planar triangulation such that

- The out-degree of each internal vertex is 3
- The out-degree of the 3 external vertices is 0
\(\alpha\)-orientations

- (\(\alpha\)-orientation) Given \(G = (V,E)\) and \(\alpha: V \rightarrow \mathbb{Z}^+\), vertex \(v\) has out-degree \(\alpha(v)\)
- Bipartite perfect matchings, Eulerian orientations etc. are special instances of \(\alpha\)-orientations
- Counting \(\alpha\)-orientations is \#P-complete
  - Let \(\alpha(v) = d(v)/2\) and then \(\alpha\)-orientations correspond to Eulerian orientations
  - Counting Eulerian orientations \#P-complete [Mihail, Winkler]
  - Still \#P-complete when restrict to planar graphs [Creed]
A Bijection with Schnyder Woods

• Each 3-orientation gives rise to a unique edge coloring known as a Schnyder wood where for each internal vertex \( v \):
  – \( v \) has out-degree 1 in each of the 3 colors
  – the clockwise order of the edges incident to \( v \) is: outgoing green, incoming blue, outgoing red, incoming green, outgoing blue and incoming red

• Schnyder woods are used in
  – Graph drawing
  – Poset dimension theory
  – Counting Planar Maps
  – And many more ….
For each internal vertex $v$:

- $v$ has out-degree 1 in each of the 3 colors
- the clockwise order of the edges incident to $v$ is: outgoing green, incoming blue, outgoing red, incoming green, outgoing blue and incoming red
Two Sampling Problems

Sample from the set of all 3-orientations of a **fixed triangulation**.

Sample from the set of all 3-orientations of triangulations with **n internal vertices**.
The Mixing Time

**Definition:** The total variation distance is

\[ ||P^t, \pi|| = \max_{x \in \Omega} \frac{1}{2} \sum_{y \in \Omega} |P^t(x, y) - \pi(x)|. \]

**Definition:** Given \( \varepsilon \), the mixing time is

\[ \tau(\varepsilon) = \min \{ t : ||P^{t'}, \pi|| < \varepsilon, \; \forall \; t' \geq t \}. \]
The Mixing Time

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A Markov chain is **rapidly mixing** if $\tau(\varepsilon)$ is $\text{poly}(n, \log(\varepsilon^{-1}))$. (or polynomially mixing)
The Mixing Time

Definition: The total variation distance is
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Definition: Given \( \varepsilon \), the mixing time is
\[ \tau(\varepsilon) = \min \{ t : \| P_{t'}^*, \pi \| < \varepsilon, \forall t' \geq t \}. \]

A Markov chain is rapidly mixing if \( \tau(\varepsilon) \) is \( \text{poly}(n, \log(\varepsilon^{-1})) \).
(or polynomially mixing)

A Markov chain is slowly mixing if \( \tau(\varepsilon) \) is at least \( \exp(n) \).
Our Results

Sample from the set of all 3-orientations of a fixed triangulation.

1. The local chain is fast if max degree $\leq 6$.

2. The local chain can take exponential time.

Sample from the set of all 3-orientations of triangulations with $n$ internal vertices.

3. The local chain is fast.
Sample from the set of all 3-orientations of a fixed triangulation.

1. The local chain is fast if max degree ≤ 6.
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Sample from the set of all 3-orientations of triangulations with n internal vertices.

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How Big is the Set of all 3-orientations of a Fixed Triangulation?
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All 3-orientations of a Fixed Triangulation

• No known efficient method for exactly counting
• There is a known FPRAS based on a bijection with perfect matchings in bipartite graphs $O^*(n^7)$ [Bezáková et al.]
  – Improving on result by [Jerrum, Sinclair, Vigoda]
  – Implies an efficient sampling algorithm exists
• Special Case: triangular lattice $O(n^4)$ algorithm using a “tower chain” [Creed]
Repeat:

- Pick a triangle $t$;
- If $t$ is a directed cycle, reverse it with probability $\frac{1}{2}$. 
The Local Markov Chain $\mathcal{M}_{TR}$

**Thm:** The local Markov chain $\mathcal{M}_{TR}$ connects the set of all 3-orientations of a fixed triangulations [Brehm].

Repeat:

- Pick a triangle $t$;
- If $t$ is a directed cycle, reverse it with probability $\frac{1}{2}$. 

![Diagram showing the process of the Markov chain]
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Let $\Delta_i(T)$ be the maximum degree of any internal vertex of $T$.

**Thm:** If $\Delta_i(T) \leq 6$ then $\mathcal{M}_{TR}$ mixes in time $O(n^8)$

[M., Randall, Streib, Tetali]
Let $\Delta_I(T)$ be the maximum degree of any internal vertex of $T$.

**Thm:** If $\Delta_I(T) \leq 6$ then $\mathcal{M}_{TR}$ mixes in time $O(n^8)$.

**Proof sketch:**

A. Define auxiliary Markov chain $\mathcal{M}_{CR}$
B. Show $\mathcal{M}_{CR}$ is rapidly mixing
C. Compare the mixing times of $\mathcal{M}_{TR}$ and $\mathcal{M}_{CR}$
Thm: If $\Delta_i(T) \leq 6$ then $\mathcal{M}_{TR}$ mixes in time $O(n^8)$ [M., Randall, Streib, Tetali]

Proof sketch:

A. Define auxiliary Markov chain $\mathcal{M}_{CR}$
B. Show $\mathcal{M}_{CR}$ is rapidly mixing
C. Compare the mixing times of $\mathcal{M}_{TR}$ and $\mathcal{M}_{CR}$

$\mathcal{M}_{CR}$ can reverse directed cycles which contain more than one triangle

- Maintains the same stationary distribution
- Moves are based on "tower moves" introduced by [Luby, Randall, Sinclair]
Repeat:
- Pick a triangle \( f \);
- If \( f \) is the beginning of a tower of length 1, reverse it with probability \( \frac{1}{2} \).
- If \( f \) is the beginning of a tower of length \( k \geq 2 \), reverse it with probability \( \frac{1}{6k} \).
A tower of length $k$ is a path of faces $f_1, f_2, \ldots, f_k$ such that:

1. $f_k$ is the only face bounded by a directed cycle
2. For every $1 \leq i < k$, the disagree edge of $f_i$ is also incident with $f_{i+1}$
3. Every vertex $v$ is incident to at most 3 consecutive faces in the path
Our Results

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Sample from the set of all 3-orientations of triangulations with $n$ internal vertices.

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**Our Results**

Sample from the set of all 3-orientations of a **fixed triangulation**.

1. The local chain $\mathcal{M}_\text{TR}$ is fast if max degree $\leq 6$.

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Sample from the set of all 3-orientations of triangulations with n **internal vertices**.

3. The local chain is fast.
Thm: There exists a triangulation $T$ on which $M_{TR}$ requires exponential time. [M., Randall, Streib, Tetali]
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A. Define a triangulation $T$
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Thm: There exists a triangulation $T$ on which $\mathcal{M}_{TR}$ requires exponential time. [M., Randall, Streib, Tetali]

Proof sketch:
A. Define a triangulation $T$
B. Show that $T$ has a “bottleneck”
There is only one 3-orientation of $T$ with edge $(v_0, v_1)$ colored green!
The “Bottleneck
The "Bottleneck

Edge \((v_0, v_1)\) colored red.
The “Bottleneck

Edge \( (v_0, v_1) \) colored red.

Edge \( (v_0, v_1) \) colored blue.
The “Bottleneck

Edge \((v_0, v_1)\) colored red.

Edge \((v_0, v_1)\) colored green.

Edge \((v_0, v_1)\) colored blue.
Two Sampling Problems

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Two Sampling Problems

Sample from the set of all 3-orientations of a **fixed triangulation**.

Sample from the set of all 3-orientations of triangulations with **n internal vertices**.
• In bijection with pairs of non-crossing Dyck paths
  (string with equal # of 1’s and -1’s where the sum is always greater than 0)

• Has size $C_{n+2}C_n - C_{n+1}^2$
  ($C_n$ is the nth Catalan number)

• Can sample using the reduction to counting
  (Explicitly worked out by [Bonichon, Mosbah])
The Local Markov Chain $\mathcal{M}_{EF}$

Repeat:

- Pick two adjacent triangles $t_1$ and $t_2$ with shared edge $xy$;
- Pick an edge $\overrightarrow{zx}$ from $t_1 \cup t_2$, if possible, replace $(\overrightarrow{zx}, \overrightarrow{xy})$ by $(xz, zw)$ with probability $\frac{1}{2}$.
The Local Markov Chain $\mathcal{M}_{EF}$

Repeat:

- Pick two adjacent triangles $t_1$ and $t_2$ with shared edge $xy$;
- Pick an edge $zx$ from $t_1 \cup t_2$, if possible, replace $(zx, xy)$ by $(xz, zw)$ with probability $\frac{1}{2}$.

**Thm:** The local Markov chain $\mathcal{M}_{EF}$ connects the set of all 3-orientations with $n$ internal vertices [Bonichon, Le Saec, Mosbah].
Our Results

Sample from the set of all 3-orientations of a fixed triangulation.

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Sample from the set of all 3-orientations of triangulations with $n$ internal vertices.

3. The local chain $\mathcal{M}_{EF}$ is fast.
Our Results

Sample from the set of all 3-orientations of a **fixed triangulation**.

1. The local chain $\mathcal{M}_{TR}$ is fast if max degree $\leq 6$.

2. The local chain $\mathcal{M}_{TR}$ can take exponential time.

Sample from the set of all 3-orientations of triangulations with $n$ **internal vertices**.

3. The local chain $\mathcal{M}_{EF}$ is fast.
Thm: $\mathcal{M}_{EF}$ mixes rapidly. [M., Randall, Streib, Tetali]
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Proof sketch:

A. Let $\mathcal{M}_{DK}$ be the “mountain to valley” chain on Dyck paths
B. $\mathcal{M}_{DK}$ is known to mix rapidly [Wilson]
C. Compare the mixing times of $\mathcal{M}_{EF}$ and $\mathcal{M}_{DK}$
Thm: $\mathcal{M}_{\text{EF}}$ mixes rapidly. [M., Randall, Streib, Tetali]

Proof sketch:

A. Let $\mathcal{M}_{\text{DK}}$ be the “mountain to valley” chain on Dyck paths
B. $\mathcal{M}_{\text{DK}}$ is known to mix rapidly [Wilson]
C. Compare the mixing times of $\mathcal{M}_{\text{EF}}$ and $\mathcal{M}_{\text{DK}}$

Although $\mathcal{M}_{\text{DK}}$ is local in the setting of Dyck paths, in the context of 3-orientations it can make global changes in a single step.
The Bijection with Dyck Paths

**Bottom Path:**
1. Trace around the blue tree in a clockwise direction
2. Add a 1 if the edge is the opposite of the trace direction
3. Otherwise add a 0

**Top Path:**
1. Let $v_1, v_2, \ldots, v_n$ be the order the vertices are encountered in step 1 above
2. Let $d_i$ be the # of incoming red edges to vertex $v_i$
3. Let $r$ be the # of incoming red edges to $s_{\text{red}}$
4. The top path is: $1(-1)^{d_2}, 1, (-1)^{d_3}, \ldots 1(-1)^{d_n} (-1)^r$
The “Mountain to Valley” chain $\mathcal{M}_{DK}$

Repeat:

- Pick $v$ on one of the paths;
- If $v$ marks a mountain/valley, invert with probability $\frac{1}{2}$, if possible.
Open Problems

1. Can the fast mixing proof for $\mathcal{M}_{\text{TR}}$ be extended to triangulations with $\Delta_1(T) > 6$?

2. Is there a family of triangulations with bounded degree where the mixing time of $\mathcal{M}_{\text{TR}}$ is exponentially large but has bounded degree?
   - Recently [Felsner, Heldt] created a, somewhat simpler, family of graphs based on our construction but the maximum degree still grows with $n$.

3. Is there an alternative local chain which can sample efficiently from the set of 3-orientations of a fixed triangulation without recourse to the bipartite perfect matching sampler of [Bezáková et al.]?
Thank you!