We proved that for every graph, every min cut $(S^*, \overline{S^*})$

$$\Pr\left(\begin{array}{c}\text{Karger's alg.}\\ \text{finds } (S^*, \overline{S^*})\end{array}\right) \geq \frac{1}{\binom{n}{2}}$$

That is true for every cut of minimum size.

So what is the maximum # of cuts of minimum size that a graph can have?

Say there are ℓ min cuts for a graph G.

Let $(S_1, \overline{S_1}), (S_2, \overline{S_2}), \ldots, (S_\ell, \overline{S_\ell})$ denote these min cuts.

Let $\mathcal{E}_i$ = denote the event that Karger's algorithm outputs cut $(S_i, \overline{S_i})$

We know that for all i,

$$\Pr(\mathcal{E}_i) \geq 1/\binom{n}{2}$$

Therefore, $\Pr(\mathcal{E}_1) + \Pr(\mathcal{E}_2) + \cdots + \Pr(\mathcal{E}_\ell) \geq \ell/\binom{n}{2}$

Only 1 cut is output so $Pr(\mathcal{E}_i \cap \mathcal{E}_j) = 0$

Thus, since these events are disjoint,

$$\underbrace{Pr(\mathcal{E}_1 \cup \mathcal{E}_2 \cup \cdots \cup \mathcal{E}_\ell)}_{\substack{\leq 1 \\ \text{Since it's a} \\ \text{probability}}} = \underbrace{Pr(\mathcal{E}_1) + Pr(\mathcal{E}_2) + \cdots + Pr(\mathcal{E}_\ell)}_{\substack{\cancel{\leq} \geq \ell / \binom{n}{2} \\ \text{as we pointed out} \\ \text{earlier}}}$$

Hence, $$1 \geq \frac{\ell}{\binom{n}{2}}$$

Therefore, $$\ell \leq \binom{n}{2}$$

So every graph has $\leq \binom{n}{2}$ cuts of minimum size.

In contrast, for min st-cuts there can be an exponential # of cuts of minimum size — see HW 1.

<u>Min-cut</u>: Given a graph $G=(V,E)$ find a cut of <u>minimum</u> size.

<u>Max-cut</u>: Given $G=(V,E)$ find a cut of <u>maximum</u> size

Min-cut $\in P$

Max-cut (decision version) is NP-complete.

In fact, NP-hard to approximate Max-cut within a factor better than $\frac{16}{17} \approx .941$ times optimal.

We'll see an easy way to find a cut within a factor $\frac{1}{2}$ of optimal.

First we'll show that every graph has a large cut.

Lemma: For a graph $G=(V,E)$ with $n=|V|$, $m=|E|$, there is a cut of size $\geq \frac{m}{2}$.

In other words, for every graph, there is a partition of the vertices V into $S, \bar{S}$ where $\geq \frac{m}{2}$ edges cross between $S \leftrightarrow \bar{S}$.

Proof:

We'll construct S randomly.
For every vertex v, independently,
add v into S with probability $\frac{1}{2}$
& into $\bar{S}$ with probability $\frac{1}{2}$.

Let $X = \#$ of edges crossing $S \leftrightarrow \bar{S}$
X is a random variable.

Look at its expectation:

$$E[X] = \sum_{j=1}^{m} j \Pr(X=j)$$

Would like an easier way to compute $E[X]$.

Let $E = \{e_1, e_2, \ldots, e_m\}$.

Let $X_i = \begin{cases} 1 & \text{if edge } e_i \text{ crosses } S \leftrightarrow \bar{S} \\ 0 & \text{otherwise} \end{cases}$

Hence, $X = \sum_{i=1}^{m} X_i$

Say $e_i = (v, w)$,
then $X_i = 1$ if $\underbrace{v \in S, w \notin S \text{ or } v \notin S, w \in S}_{}$,

fix where v lands (in S or $\bar{S}$)
then with prob. $\frac{1}{2}$ w lands in the same set
& with $\frac{1}{2}$ w lands in the other set.

Hence, $\Pr(X_i = 1) = \frac{1}{2}$.

$$E[X_i] = (1)\Pr(X_i = 1) + (0)\Pr(X_i = 0) = \frac{1}{2}.$$

$$E[X] = E\Big[\sum_j X_j\Big] = \sum_j E[X_j]$$ by linearity of expectation.

$$= \frac{m}{2}$$

So $E[X] = \frac{m}{2}$.

There must be at least one cut with size $\geq E[X]$.

Let ℓ = size of max cut in a graph G.

Suppose $\ell \leq \frac{m}{2} - 1$.

$$E[X] = \sum_j j \Pr(X=j)$$

$$= \sum_{j \leq \frac{m}{2}-1} j \Pr(X=j) + \sum_{j \geq m/2} j \Pr(X=j)$$

Since the max-cut is of size $\leq \frac{m}{2} - 1$, then for $j \geq \frac{m}{2}$, $\Pr(X=j) = 0$ (otherwise we have a cut of size $\geq \frac{m}{2}$)

$$\leq \left(\frac{m}{2} - 1\right) \Pr\left(X \leq \frac{m}{2} - 1\right) + m \underbrace{\Pr\left(X \geq \frac{m}{2}\right)}_{\to 0}$$

$$\leq \frac{m}{2} - 1.$$

So if $\ell =$ max cut size is $\leq \frac{m}{2} - 1$

then $E[X] \leq \frac{m}{2} - 1$.

But $E[X] = \frac{m}{2}$ so this is a contradiction.

Hence, it is not true that $\ell \leq \frac{m}{2} - 1$,

so $\ell > \frac{m}{2} - 1$, i.e., $\ell \geq \frac{m}{2}$.

$\boxtimes$

Another way to view it is that the random variable X is the size of a cut for a random partition $(S, \bar{S})$.

Since $E[X] = \frac{m}{2}$ there must be at least one partition where its size is $\geq \frac{m}{2}$.

SAT:

input: Boolean formula f with n variables $x_1, \dots, x_n$ & m clauses $C_1, \dots, C_m$ in CNF (conjunctive normal form)

output: satisfying assignment, if one exists

Max-SAT:

input: Same as SAT

output: assignment that maximizes the # of satisfied clauses.

Max-SAT (decision version) is NP-hard. & NP-hard within some constant factor (as with Max-cut)

Lemma: For a CNF formula F with m clauses there is an assignment satisfying $\geq \frac{m}{2}$ clauses.

Proof: Fix a formula F.

Make a random assignment.

Independently for ~~&~~ each variable x_i,

set $x_i = T$ with prob. $1/2$

& $x_i = F$ with prob. $1/2$.

Let Y = # of satisfied clauses

$$E[Y] = \sum_{j=1}^{m} j \Pr(Y=j)$$

For the i^{th} clause, let

$$Y_i = \begin{cases} 1 & \text{if the } i^{th} \text{ clause is satisfied} \\ 0 & \text{otherwise} \end{cases}$$

Then, $Y = \sum_{i=1}^{m} Y_i$

and $E[Y] = E\left[\sum_{i=1}^{m} Y_i\right] = \sum_{i=1}^{m} E[Y_i]$

$$E[Y_i] = 1 \times Pr(Y_i = 1) + 0 \times Pr(Y_i = 0)$$
$$= Pr(Y_i = 1)$$

Hence,

$$E[Y] = \sum_{i=1}^{m} Pr(Y_i = 1)$$

What's $Pr(Y_i = 1)$?

Say the i^{th} clause has k literals, then ~~it's~~ there's 1 assignment where it's unsatisfied & there are 2^k total assigments.

$$\Rightarrow Pr(Y_i = 0) = 1/2^k$$

Hence, $\Pr(Y_i = 1) = 1 - 2^{-k}$

Since $k \geq 1$,

$$\Pr(Y_i = 1) \geq 1/2.$$

Therefore,

$$E[Y] \geq \frac{m}{2}$$

& so there must be at least one assignment satisfying at least $\frac{m}{2}$ clauses.

$\boxtimes$

Linearity of expectation: random variables X & Y:

$$E[X+Y] = \sum_i \sum_j (i+j) \Pr(X=i, Y=j)$$

$$= \sum_i i \sum_j \Pr(X=i, Y=j)$$

$$+ \sum_j j \sum_i \Pr(X=i, Y=j)$$

$$= \sum_i i \Pr(X=i) + \sum_j j \Pr(Y=j)$$

$$= E[X] + E[Y].$$