Example from last class:

$$\max \quad x_1 + 6x_2 + 10x_3$$

$$\text{s.t.} \quad x_1 \le 300 \quad ①$$
$$x_2 \le 200 \quad ②$$
$$x_1 + 3x_2 + 2x_3 \le 1000 \quad ③$$
$$x_2 + 3x_3 \le 500 \quad ④$$
$$x_1, x_2, x_3 \ge 0 \quad 5a, 5b, 5c$$

Optimal is at $(x_1, x_2, x_3) = (200, 200, 100)$
which has objective value $= 2400$

How do we know it's optimal?

Let $(y_1, y_2, y_3, y_4) = (0, \frac{1}{3}, 1, \frac{8}{3})$

Then, $y_1 \times ① + y_2 \times ② + y_3 \times ③ + y_4 \times ④$
is equivalent to:

$$x_1 + 6x_2 + 10x_3 \le 2400$$

So 2400 is the max objective value

For this LP for general $y = (y_1, y_2, y_3, y_4)$,

$y_1 \times ① + y_2 \times ② + y_3 \times ③ + y_4 \times ④$

$$\Leftrightarrow x_1 y_1 + x_2 y_2 + x_1 y_3 + 3x_2 y_3 + 2x_3 y_3 + x_2 y_4 + 3x_3 y_4 \leq 300y_1 + 200y_2 + 1000y_3 + 500y_4$$

$$\Leftrightarrow \underbrace{x_1(y_1 + y_3) + x_2(y_2 + 3y_3 + y_4) + x_3(2y_3 + 3y_4)}_{} \leq \underbrace{300y_1 + 200y_2 + 1000y_3 + 500y_4}_{}$$

goal is to minimize this quantity

while LHS, is $\geq x_1 + 6x_2 + 10x_3$

which means:

$y_1 + y_3 \geq 1$

$y_2 + 3y_3 + y_4 \geq 6$

$2y_3 + 3y_4 \geq 10$

Dual LP: $\min\ 300y_1 + 200y_2 + 1000y_3 + 500y_4$

s.t.

$$y_1 + y_3 \geq 1$$
$$y_2 + 3y_3 + y_4 \geq 6$$
$$2y_3 + 3y_4 \geq 10$$
$$y_1, y_2, y_3, y_4 \geq 0$$

We had LP with $c = \begin{pmatrix} 1 \\ 6 \\ 10 \end{pmatrix}$, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 3 & 2 \\ 0 & 1 & 3 \end{bmatrix}$, $b = \begin{pmatrix} 300 \\ 200 \\ 1000 \\ 500 \end{pmatrix}$

variables $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Primal LP: $\max c^T x$

$$Ax \leq b$$

$$x \geq 0$$

Dual LP: variables $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$

$$\min b^T y$$

$$A^T y \geq c$$

$$y \geq 0$$

How to convert a LP into generic form?

- for constraint:

$$\sum_{i=1}^{n} a_i x_i \geq b \implies \sum_{i=1}^{n} (-a_i) x_i \leq -b$$

$$\sum_{i=1}^{n} a_i x_i = b \implies \begin{matrix} \sum_{i=1}^{n} a_i x_i \leq b \\ \sum_{i=1}^{n} a_i x_i \geq b \end{matrix} \implies \begin{matrix} \sum_{i=1}^{n} a_i x_i \leq b \\ \sum_{i=1}^{n} (-a_i) x_i \leq -b \end{matrix}$$

- for objective function: $\min c^T x$
  replace with $\max (-c)^T x$

- for $x_i$ that are unrestricted (so no $x_i \geq 0$ constraint)
  add variables $x_i^+$ & $x_i^-$
  & replace $x_i$ with $(x_i^+ - x_i^-)$
  & add constraints $x_i^+ \geq 0, x_i^- \geq 0$

Dual LP is $\min b^T y$

$$A^T y \geq c$$
$$y \geq 0$$

Converting into generic form we have:

$$\max (-b)^T y$$
$$(-A)^T y \leq -c$$
$$y \geq 0$$

taking its dual we have:

$$\min (-c)^T x$$
$$-Ax \geq -b$$
$$x \geq 0$$

which is equivalent to:

$$\max c^T x$$
$$Ax \leq b$$
$$x \geq 0$$

which is the original LP.

So, Dual(dual) = primal.

For a LP, optimum is achieved at a vertex of the feasible region except if:

a) it's infeasible

(example: $x_1 \leq 100$ & $x_1 \geq 200$ so feasible region is empty)

b) it's unbounded

(example: max $x_1 + x_2$

$x_1, x_2 \geq 0$)

## (Strong) Duality theorem:

If the primal LP has an optimal solution $x^*$ then the dual LP has an optimal solution $y^*$

where: $c^T x^* = b^T y^*$

(so objective functions match)

And if primal LP is unbounded then the dual LP is infeasible

if Dual LP is unbounded then the primal LP is infeasible

(could be that both primal & dual LPs are infeasible)

## Weak duality theorem:

For a LP whose primal and dual are feasible, then the optimum $x^*$ & $y^*$ satisfy:

$$c^T x^* \leq b^T y^*$$

Proof:

We'll show that for any feasible point $x$ of the primal LP and any feasible point $y$ of the dual LP, $c^T x \leq b^T y$. The theorem follows from this.

Recall, $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{i1} & \cdots & a_{in} \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ $A^T = \begin{bmatrix} a_{11} & & a_{m1} \\ a_{1i} & \cdots & a_{mi} \\ a_{1n} & & a_{mn} \end{bmatrix}$

$b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$ $c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$

Dual LP says: $A^T y \geq c$

row $i$ says: $\sum_{j=1}^{m} a_{ji} y_j \geq c_i$ (*)

Primal LP says: $Ax \le b$

row $j$ says: $\sum_{i=1}^{n} a_{ji} x_i \le b_j$ $\quad (**)$

Consider a point $x$ in the feasible region of the primal LP.
& $y$ in the " " of the dual LP.

$$c^T x = \sum_{i=1}^{n} c_i x_i$$

Since $c_i \le \sum_{j=1}^{m} a_{ji} y_j$ by $(*)$

$$c^T x = \sum_{i=1}^{n} c_i x_i \le \sum_{i=1}^{n} \left( \sum_{j=1}^{m} a_{ji} y_j \right) x_i$$

$$= \sum_{i,j} a_{ji} y_j x_i$$

$$= \sum_{j=1}^{m} \underbrace{\left( \sum_{i=1}^{n} a_{ji} x_i \right)}_{\le b_j \text{ by } (**)} y_j$$

$$\le \sum_{j=1}^{m} b_j y_j$$

$$= b^T y.$$

$\boxtimes$

Consequence of weak duality:

if we have a feasible $x$ of the primal LP

& feasible $y$ of the dual LP,

and $c^T x = b^T y$

then $x$ & $y$ are optimal solutions to the primal LP & dual LP respectively.

# Max-flow Problem:

Given a directed graph $G=(V,E)$ with two vertices distinguished $s \in V$ & $t \in V$.
Each edge has a capacity $c_e > 0$ for $e \in E$.

We want to send as much <u>flow</u> as possible from $s$ to $t$.

## Constraints:

- There's unlimited supply at $s$ & unlimited demand at $t$.
- For every edge $e \in E$, need to satisfy the edge constraints:

$$0 \leq f_e \leq c_e$$

where $f_e$ = flow along edge $e$.

- For every vertex $v \in V - \{s\} - \{t\}$,

flow-in to $v$ = flow-out of $v$,

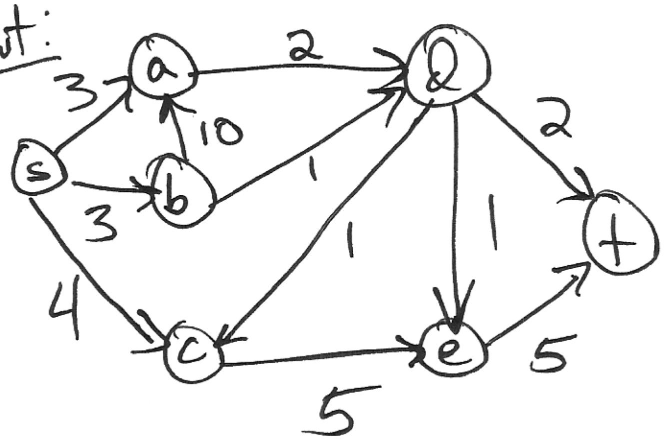
$$\text{i.e., } \sum_{\overrightarrow{wv} \in E} f_{wv} = \sum_{\overrightarrow{vz} \in E} f_{vz}$$

Finally our goal is to maximize

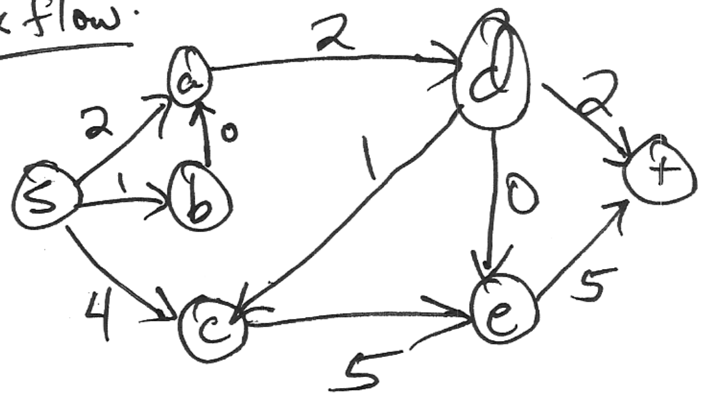
$$\text{size}(f) = \sum_{\overrightarrow{sz} \in E} f_{sz} = \text{flow out of } s = \sum_{\overrightarrow{wt} \in E} f_{wt} = \text{flow-in to } t$$

Example: (from [DPV, figure 7.4])

input:




max flow:




Can solve using LP:

Variable $f_e$ for each edge $e \in E$

objective function: $\max \sum_{\overrightarrow{sv} \in E} f_{sv}$

constraints:

for all $e \in E$, $0 \le f_e \le c_e$

for all $v \in V - s - t$,

$$\sum_{\overrightarrow{wv} \in E} f_{wv} = \sum_{\overrightarrow{vz} \in E} f_{vz}$$