Max-flow problem: Monday 9/29/14 ①

Input: Directed graph $G=(V,E)$
with specified $s\in V$ & $t\in V$
and for every edge $e\in E$ there is capacity $c_e>$

Output: Flow f specified by for every
edge $e\in E$ f_e
which maximizes
$$\text{size}(f) = \sum_{w:\overrightarrow{sw}\in E} f_{sw} = \text{flow-out of } s$$

to be a valid flow need that:

- for all $e\in E$: $0\le f_e \le c_e$
- for all $v\in V-\{s,t\}$:
  flow-in to v = flow-out of v

i.e., $$\sum_{u:\overrightarrow{uv}\in E} f_{uv} = \sum_{w:\overrightarrow{vw}\in E} f_{vw}$$

We saw last class how to solve max-flow using LP.

Min st-cut:

input: Same as for max-flow

A st-cut is a partition $V = L \cup R$ where:

$$s \in L, \; t \in R$$

The capacity of the st-cut (L,R) is

$$\text{capacity}(L,R) = \sum_{\substack{\vec{vw} \in E: \\ v \in L, w \in R}} c_{vw} = \begin{array}{c}\text{total capacity} \\ \text{of edges from} \\ L \text{ to } R\end{array}$$

output: st-cut (L,R) with minimum capacity.

Last class we saw LP duality:

Primal LP:

n variables $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

m constraints specified by $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$

$$\max \; c^T x$$
$$Ax \leq b$$
$$x \geq 0$$

Dual LP:

m variables $y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$ (one per constraint in primal LP)

n constraints specified by A^T

$$\min \; b^T y$$
$$A^T y \geq c$$
$$y \geq 0$$

Max-flow LP:

Variable f_e for each edge $e \in E$

$$\max \sum_{w:\overrightarrow{sw} \in E} f_{sw}$$

subject to:

- for all $e \in E$: $f_e \le c_e$
- for all $v \in V - \{s,t\}$: $\displaystyle\sum_{u:\overrightarrow{uv} \in E} f_{uv} = \sum_{w:\overrightarrow{vw} \in E} f_{vw}$
- for all $e \in E$: $f_e \ge 0$

We'll consider an equivalent LP.
Equivalent in the sense that it's objective value still equals the max-flow size.
But this is a HUGE LP — too many variables — So can't implement it. But we can use it to study Properties of the max-flow Problem.

The flow-preserving constraints (flow-in to v = flow-out of v) ensure no flow is "lost" along a path from s to t.

We'll avoid these constraints by specifying flow along the entire st-path.

Let $\mathcal{P}$ = set of all paths from s to t.

Typically, $|\mathcal{P}|$ is exponentially large.

For each path $p \in \mathcal{P}$, have a variable x_p.

New LP:

$$\max \sum_{p \in \mathcal{P}} x_p$$

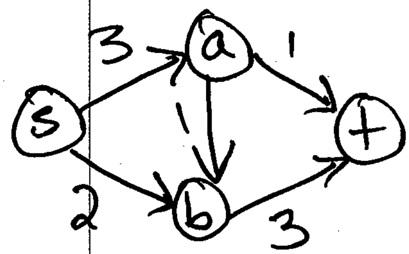
subject to:

- for all $e \in E$: $\displaystyle\sum_{p \in \mathcal{P}: e \in p} x_p \leq c_e$

- for all $p \in \mathcal{P}$: $x_p \geq 0$

Says: total flow over edge e $\leq c_e$

Example:



$\mathcal{P} = \{P_a, P_{ab}, P_b\}$ = correspond to paths $s \to a \to t$, $s \to a \to b \to t$, $s \to b \to t$

$$\max \quad P_a + P_{ab} + P_b$$

s.t.

$$P_a + P_{ab} \leq 3 \quad (\text{for edge } s \to a)$$

$$P_{ab} \leq 1 \quad (\text{for edge } a \to b)$$

$$P_a \leq 1 \quad (\text{for edge } a \to t)$$

$$P_b \leq 2 \quad (\text{for edge } s \to b)$$

$$P_{ab} + P_b \leq 3 \quad (\text{for edge } b \to t)$$

$$P_a, P_{ab}, P_b \geq 0$$

Example LP has: $x = \begin{pmatrix} P_a \\ P_{ab} \\ P_b \end{pmatrix}$ $c = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad b = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

Dual LP has: $y = \begin{pmatrix} y_{sa} \\ y_{ab} \\ y_{at} \\ y_{sb} \\ y_{bt} \end{pmatrix}$

$$\min \quad 3y_{sa} + y_{ab} + y_{at} + 2y_{sb} + y_{bt}$$

s.t.

$$y_{sa} + y_{at} \geq 1 \qquad (\text{edges on } P_a)$$

$$y_{sa} + y_{ab} + y_{bt} \geq 1 \qquad (\text{edges on } P_{ab})$$

$$y_{sb} + y_{bt} \geq 1 \qquad (\text{edges on } P_b)$$

$$y_{sa}, y_{ab}, y_{at}, y_{sb}, y_{bt} \geq 0$$

Dual LP:

for each edge $e \in E$, variable y_e

$$\min \sum_{e \in E} c_e y_e$$

subject to:

- for all $p \in \mathcal{P}$: $\sum_{e \in p} y_e \geq 1$
- for all $e \in E$: $y_e \geq 0$

Lemma: Consider a st-cut (L,R).
There is a feasible solution y where its objective value $=$ capacity(L,R).

Hence, the optimal value of the dual LP $\leq$ min st-cut.

By duality theorem,

$$\underset{\substack{\| \\ \text{max-flow}}}{\text{optimal of Primal LP}} = \text{optimal of dual LP} \leq \text{min st-cut}$$

Then we need to show that this is $=$,
So need to show next that

$$\text{optimal of dual LP} \geq \text{min st-cut}$$

Proof of lemma: For the st-cut (L,R),

$$\text{let } y_{vw} = \begin{cases} 1 & \text{if } v \in L, w \in R \\ 0 & \text{otherwise} \end{cases}$$

Then, $\sum_{vw \in E} c_{vw} y_{vw} =$ capacity(L,R).

& this y is feasible because:

for a path p, since $s \in L, t \in R$, $\sum_{vw \in p} y_{vw} \geq 1$

have to cross $L \to R$ at least once. ∎

<u>Lemma:</u> Consider a feasible y. There is a st-cut (L,R) where

$$\text{capacity}(L,R) \le \sum_{e \in E} c_e y_e$$

<u>Proof:</u> Think of y_{vw} as the weight for edge $v \to w$.

Now we have a graph with weights on edges.

For a vertex v, let

$d(v)$ = length of shortest path from s to v.

Note, for any u where $u \to v$ is an edge then

$$d(v) \le d(u) + y_{uv}$$

since we can get to v via u plus edge $u \to v$.

The dual constraints say: $\sum_{e \in P} y_e \ge 1$

Since every P ends at t, then

$$d(t) \ge 1.$$

Choose T randomly from the interval $[0,1)$.
Since $T<1$ we know $T<d(t)$.

Let $L=\{v: d(v)\leq T\}$ & $R=V-L$.

Since $t\notin L$ (because $T<d(t)$) & $s\in L$ (since $d(s)=0$)
then (L,R) is a st-cut.

Look at the expectation over T of the capacity(L,R):

$$E[\text{capacity}(L,R)]=\sum_{\overrightarrow{vw}\in E} c_{vw}\Pr(v\in L, w\in R)$$

$$\Pr(v\in L, w\in R)=\Pr(d(v)\leq T<d(w))$$
$$=d(w)-d(v)$$

But recall that $d(w)\leq d(v)+y_{vw}$

Hence, $\Pr(v\in L, w\in R)\leq y_{vw}$.

Therefore,

$$E[\text{capacity}(L,R)] \leq \sum_{\vec{vw} \in E} c_{vw} y_{vw}$$

And there must be at least one choice of T with $\text{capacity}(L,R) \leq E[\text{capacity}(L,R)]$.

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