Max-flow applications:

Baseball elimination:

set of n teams $\{1, 2, \ldots, n\}$

For team i, currently has w_i wins

Teams i & j have g_{ij} games remaining against each other

Goal: for a given team k, has k been eliminated or is there a way for k to have the most wins (at least as many as every other team)?

Example: 4 teams: NY, Baltimore, Toronto, Boston

Current wins: NY 92, Balt. 91, Tor. 91, Boston 90

5 games left = all $\binom{4}{2}$ pairs except NY-Boston

Can Boston finish in 1st place (tied or better)?

No. Why?

Boston can get ≤ 92 wins.

The other teams have 274 total wins now & 3 games against each other, so they'll have ≥ 277 wins.

Hence, at least one team has $\geq \frac{277}{3} > 92$ wins.

Is there an algorithm to determine if a team has been eliminated?

And if a team has been eliminated, is there always an averaging argument to prove it?

Yes to both questions.

Lemma: If team k has been eliminated then
- k can finish with $\leq m$ wins, and
- there is a subset T of teams where:

$$\underbrace{\frac{1}{|T|}\left(\sum_{j \in T} w_j + \sum_{i,j \in T} g_{ij}\right)}_{=\text{ minimum of average \# of wins for teams in } T.} > m$$

First, algorithm to solve it using max-flow.
Then prove the lemma.

Let $m = \#$ of wins team k has if it wins its remaining games.

Goal: determine an outcome for remaining games so that no team has $> m$ wins.

Create source s — wins emanate from s.

For each pair of teams i, j
 create vertex v_{ij}
  & edge $s \to v_{ij}$ with capacity g_{ij}

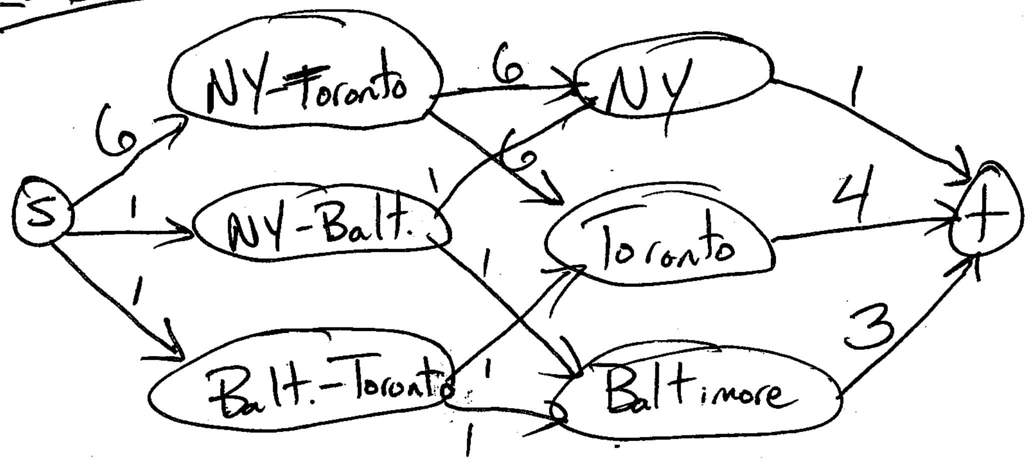
For team i,
 create vertex y_i
 for every other team j,
  add edge $v_{ij} \to y_i$
   with capacity ∞ (or anything big enough)
  & edge $y_i \to t$
   with capacity $m - w_i$ (want y_i to get $\leq m$ total wins)

Another example:

current wins: NY 90, Balt. 88, Tor. 87, Boston 79.

Games remaining: 6 with Toronto; 1 with NY, 1 with Toronto; 4 with every other team

k = BOSTON; $m = 91$



Let $g^* = \sum_{i,j \in S - \{k\}} g_{ij}$ = # of remaning games to decide

= capacity-out of s

If the max flow has value g^* then there is a way to allocate wins for all of the remaning games so that every team has $\leq m$ wins.

How do we know the max-flow has an integral solution?

Because Ford-Fulkerson only finds integral solutions, so there must exist one.

If the max-flow has value $< g^*$ then there is no way to get all teams with $\leq m$ wins so k is eliminated.

Suppose the max-flow has value $< g^*$ & so k is eliminated. Let's prove that the lemma holds.

Since max-flow has value $< g^*$ then there is a st-cut (A,B) with capacity$(A,B) < g^*$.

Let T be those teams i where $y_i \in A$. We'll prove that the lemma holds for this T.

For teams i,j where $1 \leq i,j \leq n$ & $i,j \neq k$,

Consider v_{ij}.

If ($i \notin T$ or $j \notin T$) & $v_{ij} \in A$

Then we have an edge of capacity ∞ crossing $A \to B$ so we know that if $i \notin T$ or $j \notin T$ then $v_{ij} \in B$.

Suppose $i \in T$ & $j \in T$ and $v_{ij} \in B$.

Then if we move v_{ij} to A, the cut (A,B) loses g_{ij} for edge $s \to v_{ij}$

So the capacity $(A,B) \downarrow$

Hence, since (A,B) has min capacity, if $i \in T$ & $j \in T$ we know $v_{ij} \in A$.

Therefore, $v_{ij} \in A$ iff ($i \in T$ & $j \in T$)

$\qquad\qquad\qquad\qquad\quad$ $\shortparallel$

$\qquad\qquad\qquad\qquad$ ($y_i \in A$ & $y_j \in A$)

What is capacity (A,B)?

for $i \in T$, get edge $y_i \to t$ of capacity $m - w_i$

for $v_{ij} \in B$ (so $i \notin T$ or $j \notin T$)

get edge $s \to v_{ij}$ of capacity g_{ij}

Thus,

capacity (A,B)

$$= \sum_{i \in T} (m - w_i) + \sum_{\{i,j\} \notin T} g_{ij}$$

$$= m|T| - \sum_{i \in T} w_i + \left(g^* - \sum_{i,j \in T} g_{ij} \right)$$

We know capacity $(A,B) < g^*$

Therefore, $m|T| - \sum_{i \in T} w_i - \sum_{i,j \in T} g_{ij} < 0$

$$\sum_{i \in T} w_i + \sum_{i,j \in T} g_{ij} > m|T|$$

$$\frac{1}{|T|} \left(\sum_{i \in T} w_i + \sum_{i,j \in T} g_{ij} \right) > m$$

which proves the lemma. $\boxtimes$

Image segmentation:

Given an image, separate it into objects.
Simpler setting: separate into foreground & background.

Image is on a graph $G=(V,E)$

$V=$ pixels

$E=$ neighboring pixels

(think of G as grid)

For $i \in V$,

given likelihood a_i that i is on the foreground
& b_i that i " background

$a_i \geq 0, b_i \geq 0$

for $(i,j) \in E$,

given separation penalty $p_{ij} \geq 0$.

for Partition (A,B) where $V = A \cup B$,

A = foreground

B = background

$$w(A,B) = \sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\substack{(i,j) \in E: \\ i \in A, j \in B, \\ \text{or } i \in B, j \in A}} p_{ij}$$

Goal: find partition (A,B) with max weight $w(A,B)$.

Reduce to min st-cut problem.

First, convert from maximization to minimization problem

Let $Q = \sum_{i \in V} (a_i + b_i)$

Note, $$\sum_{i\in A} a_i + \sum_{j\in B} b_j = Q - \sum_{i\in A} b_i - \sum_{j\in B} a_j$$

Then,
$$w(A,B) = Q - \sum_{i\in A} b_i - \sum_{j\in B} a_j - \sum_{\substack{(i,j)\in E:\\ i\in A,\, j\in B \text{ or}\\ i\in B,\, j\in A}} P_{ij}$$

Let $$w'(A,B) = \sum_{i\in A} b_i + \sum_{j\in B} a_j + \sum_{\substack{(i,j)\in E:\\ \dots}} P_{ij}$$

$\underset{(A,B)}{\text{Maximizing}}\ w(A,B)$

is same as

$\underset{(A,B)}{\text{minimizing}}\ w'(A,B)$

To reduce to max-flow:

for $(i,j) \in E$: add edges $i \to j$ capacity p_{ij}
& $j \to i$ capacity p_{ij}

add source s

for every $i \in V$,

add edge $s \to i$ of capacity a_i

add sink t,

for every $i \in V$,

add edge $i \to t$ of capacity b_i

Consider st-cut (A,B)

What edges cross $A \to B$?

for $j \in B$, get edge $s \to j$ of capacity a_j

for $i \in A$, get edge $i \to t$ of capacity b_i

for $(i,j) \in A$ with $i \in A, j \in B$

get $i \to j$ of p_{ij}

& if $j \in A, i \in B$

get $j \to i$ of p_{ij}

So,

$$\text{capacity}(A,B) = \sum_{i \in A} b_i + \sum_{j \in B} a_j + \sum_{\substack{(i,j) \in E: \\ i \in A, j \in B \\ \text{or } i \in B, j \in A}} p_{ij} = w'(A,B)$$

So max-flow = min st-cut & this yields the partition (A,B) with min $w'(A,B)$ & max $w(A,B)$.