Wednesday 10/15/14

Zero-sum games:

Example: Rock-Paper-Scissors

Payoff matrix:

r=rock, p=paper, s=scissors

Player 1 is row, Player 2 is column

Row \ Column	r	p	s
r	0	-1	1
p	1	0	-1
s	-1	1	0

another example:

$$G = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$$

Don't need rows or columns to sum to 0.

In general, have a $n \times n$ payoff matrix G

if row plays i & column plays j

then row gets G_{ij} & column gets $-G_{ij}$

Zero-sum game since $G_{ij} - G_{ij} = 0$.

Player 1 chooses best strategy given what they know of player 2's strategy
& player 2 chooses best strategy given what they know of player 1's strategy

Player 1 & Player 2 simultaneously announce strategies x & y respectively.

$$x = (x_1, x_2, x_3)$$
$$y = (y_1, y_2, y_3)$$

Player 1/2 plays r with prob. x_1/y_1
p with prob. x_2/y_2
s with prob. x_3/y_3

Expected payoff is

$$\sum_{i,j} G_{ij} x_i y_j$$

Player 1 wants to maximize this
& player 2 wants to minimize this.

For any fixed (or pure) strategy say for player 1
there is a strategy for player 2 to "win".

Example: suppose Row always plays scissors
then Column always plays rock

So Row should have a "mixed" strategy

<u>Two scenarios</u>:

(1) Suppose column announces his strategy y first
& then row chooses her strategy x

(2) Suppose row announces her stategy x first
& then column chooses y.

<u>Minimax theorem</u>: If both players optimize then
the value V is the same in both scenarios.

Example: for R-P-S

Suppose row plays $x = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

If column always plays Rock then the expected payoff is $\frac{1}{3} \cdot 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times -1 = 0$

Similarly, if column always plays P or S sill get expected payoff $= 0$.

So any mixed strategy gets expected payoff 0, since it's a weighted average of these 3 pure strategies.

Similarly, if column plays $y = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ then no matter what row does gives expected payoff 0.

So minimax theorem is intuitive for symmetric games.

What about nonsymmetric games?

Example:

$$G = \begin{array}{c|cc} & c & d \\ \hline a & 3 & -1 \\ b & -2 & 1 \end{array}$$

Suppose row plays $x = \left(\frac{1}{2}, \frac{1}{2}\right)$

What should column do?

if it plays c then expected payoff $= \frac{1}{2}$

if it plays d then " $= 0$

So in this case best strategy is $y = (0, 1)$

In general, for $x = (x_1, x_2)$

if it plays c, then Payoff is $= 3x_1 - 2x_2$

if it plays d, then it's $= -x_1 + x_2$

Column wants to choose the pure strategy to achieve $\min\{3x_1 - 2x_2, -x_1 + x_2\}$

In Scenario (1) — row chooses first

Row chooses $x = (x_1, x_2)$ to

maximize $\min\{3x_1 - 2x_2, -x_1 + x_2\}$

This is the following LP:

$$\max \ z$$

$$\text{s.t.} \quad z \leq 3x_1 - 2x_2$$

$$z \leq -x_1 + x_2$$

$$x_1 + x_2 = 1$$

$$x_1, x_2 \geq 0$$

For scenario (2), Column chooses $y = (y_1, y_2)$ so that minimize $\max\{3y_1 - y_2, -2y_1 + y_2\}$ which is the following LP:

$$\min \ w$$

$$\text{s.t.} \quad w \geq 3y_1 - y_2$$

$$w \geq -2y_1 + y_2$$

$$y_1 + y_2 = 1$$

$$y_1, y_2 \geq 0$$

Consider the following "relaxation" of the 1^{st} LP:

$$\max \quad z$$

$$\text{s.t.} \quad \begin{aligned} z - 3x_1 + 2x_2 &\le 0 \\ z + x_1 - x_2 &\le 0 \\ x_1 + x_2 &\le 1 \\ x_1, x_2 &\ge 0 \end{aligned}$$

(changed this constraint from $=$ to $\le$)

any feasible solution to the original LP is also feasible to this relaxed version, so the optimal value has only non-decreased (might be $=$ or might be $>$; we'll see it's $=$)

and for the 2^{nd} LP, consider:

$$\min \quad w$$

$$\text{s.t.} \quad \begin{aligned} w - 3y_1 + y_2 &\ge 0 \\ w + 2y_1 - y_2 &\ge 0 \\ y_1 + y_2 &\ge 1 \\ y_1, y_2 &\ge 0 \end{aligned}$$

Recall, for primal LP:

$$\max\ c^T x$$
$$\text{s.t. } Ax \leq b$$
$$x \geq 0$$

then Dual LP:

$$\min\ b^T y$$
$$\text{s.t. } A^T y \geq c$$
$$y \geq 0$$

Here, first LP has:

$$x = \begin{pmatrix} z \\ x_1 \\ x_2 \end{pmatrix},\quad c = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\quad A = \begin{pmatrix} 1 & -3 & 2 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

then Dual LP has 3 variables, call them y_1, y_2, w

So $y = \begin{pmatrix} y_1 \\ y_2 \\ w \end{pmatrix}$ then we have the Dual LP:

$$\min \; w$$

$$\text{s.t.} \quad y_1 + y_2 \geq 1$$

$$-3y_1 + y_2 + w \geq 0$$

$$2y_1 - y_2 + w \geq 0$$

$$y_1, y_2 \geq 0$$

this is the same as the 2^{nd} LP.

Note:

(This ignored the issue that we lacked $z \geq 0$ constraint so z is unconstrained in the primal.

Using the duality form in Figure 7.11 in [DPV] unconstrained z yields equality constraint in the dual which gives $y_1 + y_2 = 1$

& the equality constraint $x_1 + x_2 = 1$ $\Rightarrow$ unconstrained w)

In general,

Primal LP: $\max \; Z$

s.t.

for all i, $Z - \sum_j G_{ij} X_j \leq 0$

and $\sum_j X_j \leq 1$

$X_1, X_2, \ldots, X_n \geq 0$

then Dual LP is

$\min \; W$

s.t.

for all j, $W - \sum_i G_{ij} Y_i \geq 0$

and $\sum_i Y_i \geq 1$

$Y_1, \ldots, Y_n \geq 0$

Let V = objective value of optimal solution
Same for both LPs
since duals of each other

So this gives minimax theorem:

$$V = \max_{x} \min_{y} \sum_{ij} G_{ij} x_i y_j = \min_{y} \max_{x} \sum_{ij} G_{ij} x_i y_j$$

For earlier example,

$V = \frac{1}{7}$

which is achieved for $x = \left(\frac{3}{7}, \frac{4}{7}\right)$

$y = \left(\frac{2}{7}, \frac{5}{7}\right)$