undirected graph $G=(V,E)$, $n=|V|$, $m=|E|$

<u>matching</u> M is a subset of edges where each vertex is incident <u>at most</u> 1 edge of M.

<u>Perfect</u> matching P if every vertex incident to <u>exactly</u> 1 edge of M.

Hence, $|P| = \frac{n}{2}$

Maximum matching = matching of largest size

<u>Goal</u>: try to find a maximum matching

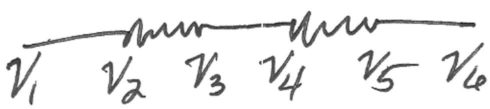
- first bipartite graphs
- then general graphs.

Consider a matching M.

Approach: augment M to get a larger matching M'
or conclude M is maximum

Alternating path: path alternating between edges in M
& edges not in M

example:



$(v_1, v_2), (v_3, v_4), (v_5, v_6)$ in M
& $(v_2, v_3), (v_4, v_5)$ not in M

or

Augmenting Path: alternating path starting & ending at unmatched vertices.

for augmenting path P,
let $M' = M \oplus P$ = flip edges along P
Note $|M'| = |M| + 1$.

Matching algorithm:

1. Start with $M = \emptyset$
2. Check if there is an augmenting path with respect to M
   - if no such path exists, output M as a maximum matching
   - if there is an augmenting path P, replace M by $M \oplus P$ & repeat

Lemma: M is maximum iff it has no augmenting paths

Proof:

($\Rightarrow$) Show that if there is an augmenting path then M is not max. This is clearly true since $M \oplus P$ is larger & is a matching

($\Leftarrow$) Assume M has no augmenting paths & let M^* be a maximum matching. We'll show $|M| = |M^*|$.

Look at $M \oplus M^*$ = edges in M or M^* & not in the other.

$M \oplus M^*$ is a collection of alternating paths & alternating cycles.

Every alternating cycle C must be even length, thus $|C \cap M| = |C \cap M^*|$

For alternating path P, if $|P|$ is odd then either: P has 1 more edge from M than from M^* so P is an augmenting path for M^*, but this contradicts that M^* is maximum.

or P has 1 more edge from M^* & then it's an augmenting path for M which contradicts our earlier assumption

So $|P|$ = even, and hence $|P \cap M| = |P \cap M^*|$.

Every component in $M \oplus M^*$ has equal # of edges from both M & M^*

So $|M| = |M^*|$. $\blacksquare$

How to find an augmenting path?

Bipartite graph $G = (L \cup R, E)$

For a matching M, orient edges of G as follows:

for $(v,w) \notin M$, $v \in L$, $w \in R$,
make it $v \to w$

for $(x,y) \in M$, $x \in L$, $y \in R$
make it $y \to x$

trying to find directed path from

~~a vertex in~~
unmatched $v \in L$ to unmatched $w \in R$

such a path must be alternating by construction.

Add vertex s with edges $s \to v$
for every $v \in L$ that is unmatched

Add vertex t with edges $w \to t$
for every $w \in R$ that is unmatched

– Run BFS from s to see if we can reach t

$O(m)$ time per BFS run, $O(n)$ runs of BFS
$\Rightarrow O(nm)$ time matching algorithm
in bipartite graphs.

Vertex cover = subset C of vertices "covering" every edge,

i.e., for $(x,y)\in E$, either: $x\in C$ &/or $y\in C$.

if C is a vertex cover
then $V-C$ is an indepent set (contains no edges)

for a matching M & vertex cover C,

for $(x,y)\in M$,

$x\in C$ &/or $y\in C$

Hence, $|M|\leq|C|$

Therefore, $\max_{M}|M|\leq\min_{C}|C|$.

max matching size $\leq$ min vertex cover size.

<u>Konig's theorem:</u> For bipartite graphs,

max matching size = min vertex cover.

Proof: For matching M, let

L^*, R^* be the unmatched vertices in L, R.

Let S be those vertices reachable in the directed graph from a vertex in L^*.

= vertices ~~reached~~ visited by the BFS from s.

Lemma: $C^* = (L-S) \cup (R \cap S)$ is a VC

Proof: For matching M^* produced by the algorithm.

let L^*, R^* be the unmatched vertices in L, R

augmenting path = directed path from a vertex in L^* to a vertex in R^*.

Let S = vertices reached in the BFS from s.
= vertices reachable from a vertex in L^*.

Let $C^* = (L - S) \cap (R \cap S)$

Lemma: C^* is a vertex cover & $|C^*| = |M^*|$
where M^* is the maximum matching produced by the earlier algorithm.

This proves Konig's theorem.

Proof of lemma:

Suppose C^* is not a vertex cover.
Then there is an edge $(x,y) \in E$ where $x \notin C^*, y \notin C^*$
Hence, $x \in L \cap S$ & $y \in R - S$.

2 Cases for edge (x, y):

1. $(x,y) \in M^*$: then edge $y \to x$ & no other incoming edges to x.
   But $x \in L \cap S$ so it's reached and the only way is from y.
   Thus, y must be in S but this contradicts $y \in R - S$.

2. $(x,y) \notin M^*$: then edge $x \to y$
   Since $x \in S$ then y is reachable from x & so $y \in S$ but this contradicts $y \in R - S$.

Thus, C^* is a vertex cover.
We know that $|M^*| \leq |C^*|$ so we just need to show $|C^*| \leq |M^*|$.

3 facts:

1. No vertex in $L-S$ is unmatched.
   Why? Because all unmatched vertices in L are connected to vertex s & thus in S.

2. No vertex in $R \cap S$ is unmatched.
   Why? Otherwise we have an augmenting path & then M^* is not the final, maximum matching produced by the algorithm.

3. No edge $(x,y) \in M$ where $x \in L-S$ & $y \in R \cap S$
   Why? otherwise the edge $y \to x$ is there, and since $y \in S$ we have $x \in S$ $\Rightarrow\Leftarrow$

Therefore, from 1 & 2, every vertex in C^* is matched, by 3 these edges are distinct.

Thus, $|C^*| \leq |M^*|$.

∎

General graphs:

Problem are odd cycles, say of size $2k+1$ vertices, with k matched edges

Example:

~~~ = matching

Called a Blossom.

Lemma: Let M be a matching of G & B be a Blossom.

Assume that B is disjoint with the rest of M (no vertices in common so v^* is unmatched)

Let G' be G with B contracted to a single vertex.

Let M' be matching M induced on G'.

Then, M' is maximum in G' iff M is maximum in G
~~~

Proof:

($\Leftarrow$) Assume M is maximum & suppose M' is not maximum in G'.

Since M' is not maximum we know there exists an augmenting path P' in G' with respect to M'.

2 cases:

- P' does not intersect contracted B.
  Then P' is also an augmenting path in G
  So M is not maximum.
  $\Rightarrow\Leftarrow$.

- P' intersects contracted B
  Since rest of M is disjoint from B
  then P' must intersect B at an unmatched vertex of P'.
  The unmatched vertices of P' are its start & end, say w is the one in B.
  Let v^* be the unmatched vertex of B.
  Let P'' be the path from w to v^* in B starting with a matched edge, & ending unmatched.
  Then, $P = P' \cup P''$ is an augmenting path in G
  So M is not a max matching
  $\Rightarrow\Leftarrow$.

($\Rightarrow$) Assume M' is maximum & suppose M is not maximum.

Take augmenting path P in G.

If P does <u>not</u> intersect B then

P is also an augmenting path in G'

So M' is not maximum. $\Rightarrow\!\Leftarrow$

If P intersects B, note:

B has 1 unmatched vertex v^*

P has 2 unmatched vertices (start & end)

Say z is the one different from v^*.

Let P' be the path starting at z, following P until hitting B.

Since B is disjoint from the rest of M, then P' is an augmenting path in G'

So again M' is not maximum.

$\Rightarrow\!\Leftarrow$

New approach:

Look for augmenting paths:

if we find a Blossom, shrink it &

continue on the smaller graph

We'll do the algorithm next time.