Undirected $G=(V,E)$, $n=|V|$, $m=|E|$.

Matching $M$ = subset of edges where each vertex is incident $\leq 1$ edge of $M$.

<u>Goal</u>: find maximum matching

for matching $M$,
alternating path $P$ = path alternating between edges in $M$ & not in $M$

augmenting path $P$ = alternating path that starts & ends at unmatched vertices.

for augmenting path $P$,
$M' = M \oplus P$ = flip edges along $P$
Note, $M'$ is a matching & $|M'| = |M| + 1$

<u>Lemma</u>: $M$ is a max matching iff it has no augmenting paths

(We proved this last class.)

## Matching algorithm:

1. Start with $M = \emptyset$
2. Check if there is an augmenting path with respect to $M$

   - if no such path exists, output $M$ as a maximum matching.
   - if there is an augmenting path $P$, replace $M$ by $M \oplus P$ & repeat

How to find an augmenting path?

Let's first do it for bipartite graphs.
(slightly different view of the algorithm from last class.)

Consider bipartite $G=(V,E)$ where $V = L \cup R$.

We'll label vertices: EVEN, ODD, or unlabeled
edges: explored or unexplored.

Since $G$ is bipartite, we'll get EVEN $\cup$ ODD as the bipartitions.

Start with all vertices unlabeled & edges unexplored.

Consider a matching $M$.

(1) Take an unlabeled & unmatched vertex $r$.
Label $r$ EVEN.
This will be the root of a new tree.

(2) Take an unexplored edge $(v,z)$ where $v$ is labeled EVEN.
Mark $(v,z)$ as explored and:

(a) if $z$ is unmatched then the path $r \rightsquigarrow z$ is an augmenting path $P$ since $r$ & $z$ are unmatched.
Output $P$ & we're done.

(b) if $z$ is unlabeled & ~~un~~matched then let $(z,y)$ be the edge in $M$.
Label $z$ as ODD & $y$ as EVEN, & mark $(z,y)$ as explored.

(c) if $z$ is already labeled ODD
then do nothing.
(we found an alternating path, but not an augmenting path since is matched)
(Note $z$ may be in a different tree.)

Repeat step (2) until no unexplored edges connected to this tree. Then go to step (1).

This finds an augmenting path or we conclude there are no augmenting paths so $M$ is maximum.

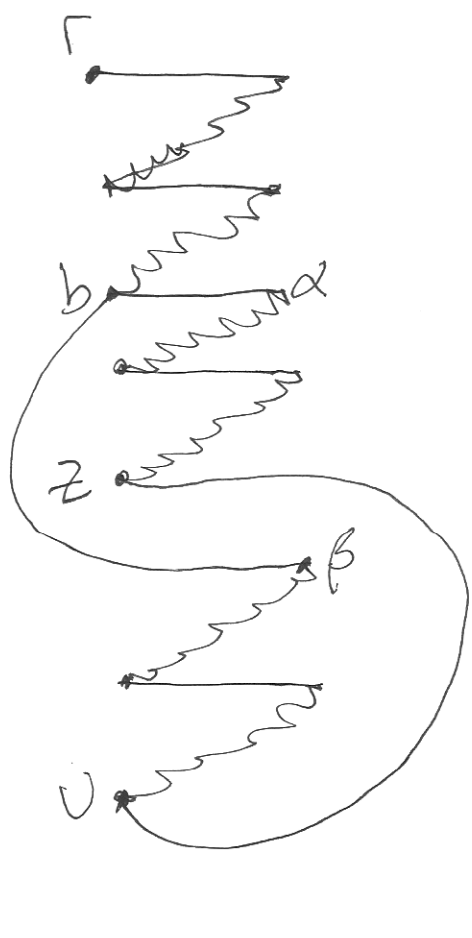
For nonbipartite graphs there are additional cases as we may have already seen $z$ & labeled it as EVEN.

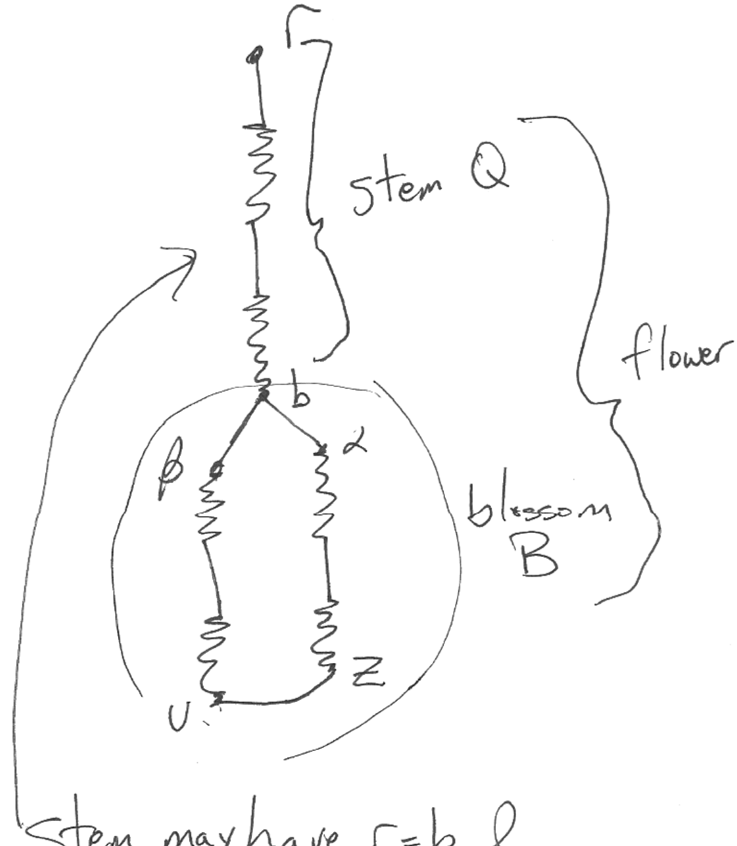
So we add two more cases:

(d) if $z$ is already labeled EVEN, then either:

- $v$ & $z$ are in different trees, but this cannot occur because we would've explored $(v,z)$ when considering $z$'s tree.
- $v$ & $z$ are in the same tree — then we have a blossom $B$.

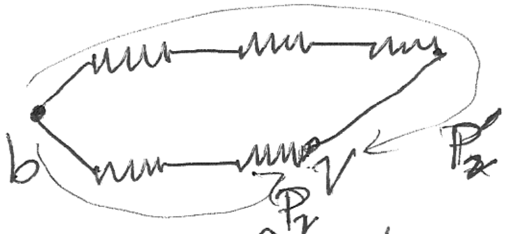
Example: ~~~~ = matched edges




Stem may have $r=b$ & then it has length 0.

Blossom $B$ = odd cycle $C$ with root $b$ where
2 edges incident to $b$ are not in $M$
& rest of cycle is alternating path.




for every $v \in C$, there is an alternating path $P_v$
$b \to v$ ending with a matched edge
& a path $P_x$ ending with
an unmatched edge

Stem $Q$ = alternating path of even length
starting at an unmatched $r$,
and ending with a matched edge.

When we find a blossom $B$,
shrink it to a new vertex $B$, call it $G_B$
Modify $M$ to $M_B = M - B$.
Run the algorithm on $G_B$ with $M_B$.

So the algorithm either finds an augmenting path,
shows that there are no augmenting paths,
or shrinks the graph by contracting a blossom.

**Lemma:** if there is an augmenting path $P$ for $M_B$ in $G_B$ then there is also an augmenting path for $M$ in $G$.

**Proof:** if $P \cap B = \emptyset$ then nothing to change

Suppose $P$ intersects $B$:

**Case 1:** $P$ starts at $B$ (or ends at $B$).

Say $P = B, c, P_1$ where $P_1$ is the rest of the path $P$.

$B$ must be unmatched in $G_B$ so the stem is length 0, i.e., $b = r$, and $b$ is unmatched in $G$.

There's at least one $v \in B$ where $(v, c) \in E$.

Let $P_v$ be the path in $B$ from $b$ to $v$ ending with a matched edge.

Then, $P_v, (v, c), P_1$ is an augmenting path in $G$.

Case 2: $B$ is not the $1^{st}$ or last vertex on $P$.

Let $(a,B),(B,c) \in P$ be the edges of $P$ touching $B$.

Assume $(a,B) \in M_B$ & $(B,c) \notin M_B$.

Let $P_a = P$ up to $a$ & $P_c = P$ from $c$ on.

The only edge entering $B$ is the edge to $b$, on the stem.

There's at least one $v \in B$ where $(v,c) \in E$.

Let $P_v$ be the path in $B$ from $b$ to $v$ ending at a matched edge

Then, $P_a, (a,b), P_v, (v,c), P_c$

is an augmenting path in $G$.

$\boxtimes$

**Lemma:** if there is an augmenting Path $P$ for $M$ in $G$, then there is also one for $M_B$ in $G_B$.

**Proof:** if $P \cap B = \emptyset$ then nothing to do.
Assume $P \cap B \neq \emptyset$.

**Case 1:** $b$ is unmatched by $M$ (so $B$ is unmatched in $G_B$)

$P$ starts & ends at an unmatched vertex.

$B$ has only one unmatched vertex (namely $b$)

So assume $P$ does not start at $b$

Let $(a,v) \in P$ where $v \in B$ be the first edge of $P$ hitting $B$.

Let $P_a = P$ up to $a$.

Then, $P_a, (a,B)$ is an augmenting Path in $G_B$.

Case 2: $b$ is matched by $M$ (so $B$ is matched in $G_B$)

following this matched edge from $b$ along $P$ we get a stem $Q$ for blossom $B$.

Let $M' = M \oplus Q$ = flip edges along $Q$.

$Q$ is even length so $|M'| = |M|$.

$B$ is still a blossom with respect to $M'$ but $b$ is unmatched in $M'$.

Then we can apply Case 1 & get an augmenting path for $M'_B$ in $G_B$.

Note $|M'_B| = |M_B|$ so if $M'_B$ has an augmenting path in $G_B$ (hence not max size) then $M_B$ also has an augmenting path in $G_B$.

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