For a $n \times n$ matrix A, its determinant is

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_i A_{i,\sigma(i)}$$

S_n = set of permutations of $\{0, 1, \ldots, n-1\}$

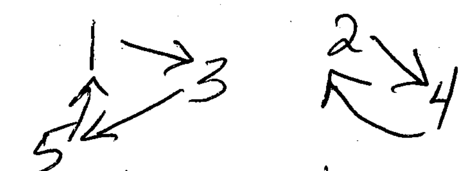
& $\operatorname{sgn}(\sigma) = (-1)^{\#\text{even cycles in } \sigma} = (-1)^{N(\sigma)}$

$$= \begin{cases} +1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$$

Parity of σ = Parity of # of inversions $N(\sigma)$

$$= \#\{x, y : x < y, \sigma(x) > \sigma(y)\}$$

Example: $\sigma = (3, 4, 5, 2, 1)$



1 even cycle

$N(\sigma) = 7$

$\operatorname{sgn}(\sigma) = -1$

Can compute the $\det(A)$ in poly-time.

Permanent:

$$\mathrm{Per}(A) = \sum_{\sigma \in S_n} \prod_i A_{i,\sigma(i)}$$

Same as determinant but drop the sign of the permutation.

Say A is a $0/1$ matrix then

$\mathrm{Per}(A) = \#$ of perfect matchings in bipartite graph G with $n+n$ vertices & edge (i,j) iff $A(i,j) = 1$

Can we compute $\mathrm{Per}(A)$ in poly-time

focus on 0-1 matrices.

For planar graphs, there is a poly-time algorithm.

We'll change G into a directed graph $\vec{G}$ in a particular way & then it'll be the case that $\mathrm{per}(A) = \mathrm{Det}(\vec{A})$ where $\vec{A}$ is adjacency matrix of $\vec{G}$.

For undirected $G=(V,E)$
an <u>orientation</u> $\vec{G}=(V,\vec{E})$ is an assignment of a direction $\overrightarrow{ij}$ or $\overrightarrow{ji}$ for every edge $(i,j)\in E$

For an even length cycle C (in undirected G) its <u>oddly oriented</u> in $\vec{G}$ if there is an odd # of clockwise edges.

Why only concerned with even length cycles?

Because:
for a pair of perfect matchings M, M'
$M \cup M'$ is a collection of disjoint even length cycles & edges (cycles of length 2)

Example:

Orientation $\vec{G}$ is Pfaffian if:
for all perfect matchings M, M'
all cycles of $M \cup M'$ are oddly oriented.

For an orientation $\vec{G}=(V,\vec{E})$ where $V=\{0,1,\dots,n-1\}$

let A be its adjacency matrix where:

$$A(i,j)=\begin{cases}1 & \text{if } \vec{ij}\in E\\ -1 & \text{if } \vec{ji}\in E\\ 0 & \text{otherwise.}\end{cases}$$

Theorem: For a Pfaffian orientation $\vec{G}$ of G,

$$\mathrm{Det}(A)=(\#\text{ of perfect matchings of } G)^2$$

For a planar graph G we can find a Pfaffian orientation in poly-time. Use the following lemma:

Lemma: For a planar graph G with orientation $\vec{G}$, if every face (except possibly the outer face) has an odd # of clockwise edges then $\vec{G}$ is a Pfaffian orientation.

Recall, for any planar drawing of $G=(V,E)$

Euler's formula: $n-e+f=1+c$

where $n=|V|$, $e=|E|$, $f=$ # faces, $c=$ # components.

Using the lemma we can construct a Pfaffian orientation.

The algorithm constructs it in a recursive/inductive manner, inducting on # of edges.

Base case: G is a tree. Then only face is the outer face so any orientation is OK.

For arbitrary planar G. Take an edge e lying on the outer face.

Look at $G-e$, & inductively find a Pfaffian orientation of it.

Adding back in e creates one new face f.
One of the two orientations for e gives the correct parity for the # of clockwise edges on f.

Proof of lemma:

Let (i) every face (except outer one) has odd # of clockwise edges

(ii) for all cycles $\vec{C}$, # of clockwise edges of $\vec{C}$ is opposite parity of # of vertices in interior of $\vec{C}$

(iii) $\vec{G}$ is a Pfaffian orientation

We'll show (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii).

First we'll do (ii) $\Rightarrow$ (iii).

Consider perfect matchings M, M' & a cycle C in $M \cup M'$.

if a vertex inside C is matched to a vertex outside C then its not planar. So # of vertices inside C must be even since they are matched with each other by M.

Hence by (ii) every cycle C in $M \cup M'$ has an ~~#~~ odd # of clockwise edges.

By defition, its Pfaffian.

Now (i) $\Rightarrow$ (ii)

Consider a cycle $\vec{C}$ of $\vec{G}$.

Look at the induced subgraph on & inside $\vec{C}$, but connected to $\vec{C}$.
so it's 1 component.

Let F_{IN} = # of faces except outer face

Let $f_1, \ldots, f_{F_{IN}}$ denote these faces

Let V_{ON} & E_{ON} denote the # of vertices & edges on $\vec{C}$.

Let V_{IN} & E_{IN} denote the # of vertices & edges strictly inside $\vec{C}$.

For face f_i, let $E_{ON}^{CLOCK}(f_i)$ & $E_{ON}^{CLOCK}(C)$ denote the # of clockwise edges on face f_i & on cycle $\vec{C}$.

Euler's formula: $(V_{ON}+V_{IN})-(E_{ON}+E_{IN})+(F_{IN}+1)=2$

Since $\vec{C}$ is a cycle, $V_{ON}=E_{ON}$

Thus, $V_{IN}-E_{IN}+F_{IN}=1$

By (i), for every face f_i,
$$E_{ON}^{CLOCK}(f_i)\equiv 1 \bmod 2.$$

Hence, $F_{IN}\equiv \sum_i E_{ON}^{CLOCK}(f_i) \bmod 2.$

Each edge e inside $\overrightarrow{C}$ is clockwise to exactly one of the 2 faces incident to e.

Each edge e' that is clockwise on C is also clockwise on the internal face containing e'.

Thus,
$$E_{IN} + E_{ON}^{CLOCK}(C) = \sum_i E_{ON}^{CLOCK}(f_i)$$

Therefore,
$$F_{IN} \equiv E_{IN} + E_{ON}^{CLOCK}(C) \mod 2$$

Recall,
$$E_{IN} = V_{IN} + F_{IN} - 1$$

Thus,
$$\cancel{F_{IN}} \equiv E_{ON}^{CLOCK}(C) + V_{IN} + \cancel{F_{IN}} - 1 \mod 2$$
$$E_{ON}^{CLOCK}(C) + V_{IN} \equiv 1 \mod 2$$

∎

Proof of Theorem: For undirected $G=(V,E)$.

Let $\overleftrightarrow{G}$ be the directed graph $\overleftrightarrow{G}=(V,\overleftrightarrow{E})$
where undirected edge $(i,j)\in E$
is replaced by directed edges $\overrightarrow{ij}$ & $\overrightarrow{ji}$.

An even cycle cover is a set of vertex disjoint
even length directed cycles containing all vertices.

Claim: $\begin{pmatrix}\#\text{ of perfect}\\ \text{matchings of } G\end{pmatrix}^2 = \#$ of even cycle covers of $\overleftrightarrow{G}$

Proof: We'll define a bijection between
pairs of perfect matchings M, M' $\longleftrightarrow$ even cycle cover

Take M, M'. Look at $M \cup M'$. For cycle C in $M \cup M'$,
orient in a canonical way: take smallest vertex v in C
take edge of M incident to v. Orient it away
from v then follow around C in same direction.

Given even cycle cover, for a cycle C can invert
the previous mapping to get a pair M, M'. ∎

Recall, $\det(A) = \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_i A(i, \sigma(i))$

For permutation, break into cycles $\sigma = \gamma_1 \cdots \gamma_k$

let V_i = vertices on cycle γ_i.

Suppose σ contains an odd length cycle.

Let γ_i be the 1st odd length cycle.

Let $\sigma' = \gamma_1 \cdots \gamma_{i-1} \gamma_i^{-1} \gamma_{i+1} \cdots \gamma_k$

reverse γ_i

σ & σ' have same # of cycles & same lengths,

So $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma')$

Since γ_i is odd length,

# of clockwise on γ_i $\neq$ # of counterclockwise mo[ves] on γ_i

Hence, $\prod_{j \in V_i} A(j, \sigma(j)) = -\prod_{j \in V_i} A(j, \sigma'(j$

Thus, in $\det(A)$, σ & σ' cancel each other out.

Take σ where all cycles are even length.

Then, $\prod_{j \in V_i} A(j, \sigma(j)) = -1$ for all i

because $\vec{G}$ is Pfaffian.

Thus,

$$\prod_{k=0}^{n-1} A(k, \sigma(k)) = \prod_i \prod_{j \in V_i} A(j, \sigma(j))$$

$$= (-1)^k \quad k = \#\text{cycles in } \sigma$$

$$= \operatorname{sgn}(\sigma)$$

Therefore,

$$\det(A) = \sum_\sigma \operatorname{sgn}(\sigma) \prod_i A(i, \sigma(i))$$

$$= \sum_{\substack{\sigma:\text{ all cycles} \\ \text{in } \sigma \text{ are} \\ \text{even length}}} (\operatorname{sgn}(\sigma))^2 = \#\text{ even cycle covers.}$$

$\boxtimes$