Given an undirected $G=(V,E)$
we saw how to determine if G has a perfect matching.

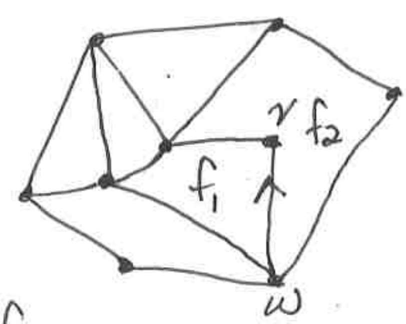
Can we determine the # of perfect matchings?
This # might be exponentially large in $n=|V|$
but want running time to be $\text{Poly}(n)$.

For planar graphs can do so. [Kasteleyn '67]

Given a graph we can determine in poly-time
if its planar & a planar embedding (if it is planar).

We'll use the planar embedding of our planar graph:

Example:



7 faces including outer face

every edge incident exactly 2 faces

for every face can go around it while keeping the face on the right side — call this clockwise direction

or on the left side — counter clockwise.

an edge is clockwise on one face
& counter clockwise on the other

Example: (v,w) is oriented $\overrightarrow{vw}$ which is
clockwise on f_2
counter clockwise on f_1.

We'll take an undirected planar graph $G=(V,E)$ and orient every edge: so $(v,w)\in E$ will be $\overrightarrow{vw}$ or $\overrightarrow{wv}$

Call this new graph $\vec{G}=(V,\vec{E})$ an orientation.

Define the adjacency matrix A of $\vec{G}$ as:

$$A(i,j)=\begin{cases} +1 & \text{if } \overrightarrow{ij}\in\vec{E} \\ -1 & \text{if } \overrightarrow{ji}\in\vec{E} \\ 0 & \text{if } (i,j)\notin E \text{ (not connected in either direction)} \end{cases}$$

We want an orientation $\vec{G}$ where for every cycle C in G which is even length there is an odd # of edges of C which are clockwise in $\vec{G}$.

(since C is even length if there are an odd # clockwise then there are an odd # counter-clockwise)

Call such an orientation a <u>Pfaffian</u> orientation.

We can find it for planar graphs.

The consequence is that for a Pfaffian orientation $\vec{G}$:

$$\#\text{ of perfect matchings in } G = \sqrt{\det(A(\vec{G}))}$$

How to find a Pfaffian orientation for planar graphs?

We'll satisfy a slightly weaker condition.

Take a pair of perfect matchings M, M'.

Look at $M \cup M'$.

Every vertex is incident one edge of M & one edge of M'.

So $M \cup M'$ consists of a set of even-length cycles (some of these cycles are size 2 in the sense that an edge $e \in M \cap M'$)

Revised Definition of Pfaffian orientation: for all cycles obtainable from a pair of perfect matchings there is an odd # of clockwise edges.

This new condition is weaker because there may be even length cycles that are not in $M \cup M'$ for any M, M' & thus there is no statement about these cycles in the new condition.

But this new condition is still sufficient for the theorem.

Back to finding an orientation satisfying this new condition, which we'll now call a Pfaffian orientation.

We'll consider an additional condition:

(*) every face (except possibly the outer face) has an odd # of clockwise edges

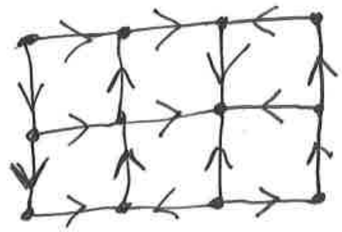
We saw last class that it's easy to recursively construct an orientation satisfying (*).

Why does (*) imply the orientation is Pfaffian?

Consider a cycle C obtainable by the union of a pair of perfect matchings M, M'.

Look at the induced subgraph on & inside C.

Example:



Let V_{ON} & E_{ON} denote # of vertices & edges on C.

Note $V_{ON} = E_{ON}$ (= 10 in this example)

Let $E_{ON}^{CLOCK}(C)$ denote # of clockwise edges on C (=3)

Let F_{IN} denote # of faces (except outer face) (=6)

& let $f_1, \ldots, f_{F_{IN}}$ denote these faces

Let $E_{ON}^{CLOCK}(f_i)$ denote # of clockwise edges on f_i

Let V_{IN} & E_{IN} denote # of vertices & edges inside C.

Euler's formula for planar graphs:

$$n - e + f = 2 \quad \text{where} \quad \begin{array}{l} n = \#\text{ vertices} \\ e = \#\text{ edges} \\ f = \#\text{ faces} \end{array}$$

Hence, $(\cancel{V_{ON}} + V_{IN}) - (\cancel{E_{ON}} + E_{IN}) + (F_{IN} + 1) = \cancel{2}$

$$V_{IN} + F_{IN} - E_{IN} = 1$$

$$E_{IN} = V_{IN} + F_{IN} - 1$$

We're assuming (*) so $E_{ON}^{CLOCK}(f_i) \equiv 1 \mod 2$ for all i.

Thus, $F_{IN} \equiv \sum_i E_{ON}^{CLOCK}(f_i) \mod 2$

Each internal edge is incident to 2 internal faces & is clockwise on exactly one of them.

& each clockwise edge on C is clockwise on its internal face.

Thus, $E_{ON}^{CLOCK}(C) + E_{IN} = \sum_i E_{ON}^{CLOCK}(f_i)$

Therefore,

$$F_{IN} \equiv E_{ON}^{CLOCK}(C) + E_{IN} \mod 2$$

Plugging in $E_{IN} = V_{IN} + F_{IN} - 1$ we have:

$$F_{IN} \equiv E_{ON}^{CLOCK}(C) + V_{IN} + F_{IN} - 1 \mod 2$$

$$E_{ON}^{CLOCK}(C) \equiv V_{IN} + 1 \mod 2$$

So the # of clockwise edges on C is opposite parity to V_{IN}.

What's the parity of V_{IN}?

C is obtained by taking $M \cup M'$ for perfect matchings M, M'.

In one of these perfect matchings, say M, all vertices in V_{IN} are matched since G is planar they can only be matched with other vertices in V_{IN}.

Thus, V_{IN} is even.

Therefore, $E_{ON}^{CLOCK}(C)$ is odd which proves that it's a Pfaffian orientation. $\blacksquare$

Proof that for Pfaffian orientation $\vec{G}$,

$$\det(A(\vec{G})) = (\#\text{ of perfect matchings})^2$$

for $n \times n$ matrix A,

$$\det(A) = \sum_{j=1}^{n} (-1)^{j+1} A(1,j) \det(A_{1j})$$

where $A(1,j) =$ is the $(1,j)$ entry of A

and A_{1j} is A with row 1 & column j removed

alternative formula:

$$\det(A) = \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} A(i, \sigma(i))$$

where $\operatorname{sgn}(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$

& σ ranges over all permutations of $\{1, \ldots, n\}$.

an equivalent form for $\operatorname{sgn}(\sigma)$ is

$$\operatorname{sgn}(\sigma) = (-1)^{\#\text{even cycles in } \sigma}$$

For any orientation $\vec{G}$ in the formula for $\det(A(\vec{G}))$ we can restrict attention to σ consisting of only even length cycles.

Why? Consider a $\sigma = \gamma_1 \gamma_2 \cdots \gamma_k$ where γ_j's are the cycles of σ.

let γ_j be the first odd length cycle (if there's more than one, choose the one with the lowest vertex)

let $\sigma' = \gamma_1 \cdots \gamma_{j-1} \gamma_j^{-1} \gamma_{j+1} \cdots \gamma_k$

So σ' is the same as σ except we flip the directions in cycle γ_j.

Note, $\mathrm{sgn}(\sigma) = \mathrm{sgn}(\sigma')$

But if σ has an odd # of edges in γ_j aligned in opposite directions of $\vec{G}$

then σ' has an even # (and if σ has even # then σ' has odd #)

So
$$\prod_{i \in \gamma_j} A(i, \sigma(i)) = -\prod_{i \in \gamma_j^{-1}} A(i, \sigma'(i))$$

Then they cancel each other out & we're left with σ's that contain only even length cycles.

Take $\sigma = \gamma_1 \cdots \gamma_k$ where all γ_j are even length

For each γ_j, $\vec{G}$ has an odd # of edges in each direction since it's Pfaffian.

Hence, $$\prod_{i \in \gamma_j} A(i, \sigma(i)) = -1$$

Therefore, $$\prod_{i=1}^{n} A(i, \sigma(i)) = (-1)^k$$ where k is the # of cycles in σ

but $\operatorname{sgn}(\sigma) = (-1)^k$ as well

thus $$\operatorname{sgn}(\sigma) \prod_{i=1}^{n} A(i, \sigma(i)) = 1$$

In conclusion, we have that:

$$\det(A(\vec{G})) = \#\text{ of } \sigma \text{ with all even length cycles}$$

Finally, one construct a bijection between

Pairs of Perfect matchings $\longleftrightarrow$ permutations σ with all even length cycles

This proves the theorem.